Proof. 首先，我们将 $\varphi(x)$ 在 $\int_X f(x)d\mu(x)$ 处进行泰勒展开

$$\varphi(x) = \varphi\left(\int_X f(x)d\mu(x)\right) + \varphi'\left(\int_X f(x)d\mu(x)\right)\left(x - \int_X f(x)d\mu(x)\right) + \frac{\varphi''(x_i)]}{2}\left(x - \int_X f(x)d\mu(x)\right)^2$$

其中 x_i 是 x 和 $\int_X f(x)d\mu(x)$ 的中间值。于是

$$\begin{aligned}\int_X \varphi \circ f(x)d\mu(x) = &\int_X \varphi\left(\int_X f(x)d\mu(x)\right)d\mu(x)\\ &+ \int_X \varphi'\left(\int_X f(x)d\mu(x)\right)\left(x - \int_X f(x)d\mu(x)\right)d\mu(x)\\ &+ \int_X \frac{\varphi''(x_i)]}{2}\left(x - \int_X f(x)d\mu(x)\right)^2 d\mu(x)\end{aligned}$$

其中

$$\int_X \varphi\left(\int_X f(x)d\mu(x)\right)d\mu(x) = \varphi\left(\int_X f(x)d\mu(x)\right)\mu(X) = \varphi\left(\int_X f(x)d\mu(x)\right)$$

$$\int_X \varphi'\left(\int_X f(x)d\mu(x)\right)\left(x - \int_X f(x)d\mu(x)\right)d\mu(x) = 0$$

因为 φ是凸函数，所以 $\varphi''(x_i) \geq 0$, 因而

$$\int_X \frac{\varphi''(x_i)]}{2}\left(x - \int_X f(x)d\mu(x)\right)^2 d\mu(x) \geq 0$$

于是得证：

$$\varphi\left(\int_X f(x)d\mu(x)\right) \leq \int_X \varphi \circ f(x)d\mu(x).$$

我们需要证明对于$p<p'$，有$m(p)\leq m(p')$，即：

$$\left[\int_X |f(x)|^p d\mu(x)\right]^{\frac{1}{p}} \leq \left[\int_X |f(x)|^{p'} d\mu(x)\right]^{\frac{1}{p'}}$$

我们可以使用 Holder 不等式来证明这个结论。设 q 是满足 $\frac{1}{p}+\frac{1}{q}=1$ 的实数，那么有

$$\begin{aligned}\int_X |f(x)|^p d\mu(x) &= \int_X |f(x)|^p \cdot 1 d\mu(x) \\ &\leq \left[\int_X |f(x)|^{p\cdot q} d\mu(x)\right]^{\frac{1}{q}} \cdot \left[\int_X 1^{q'} d\mu(x)\right]^{\frac{1}{q'}} \quad = \left[\int_X |f(x)|^{p\cdot q} d\mu(x)\right]^{\frac{1}{q}}\end{aligned}$$

其中 q' 是满足 $\frac{1}{q}+\frac{1}{q'}=1$ 的实数。同样地，应用 Holder 不等式，可得

$$\begin{aligned}\int_X |f(x)|^{p'} d\mu(x) &= \int_X |f(x)|^{p'} \cdot 1 d\mu(x) \\ &\geq \left[\int_X |f(x)|^{p'\cdot q'} d\mu(x)\right]^{\frac{1}{q'}} \cdot \left[\int_X 1^{q} d\mu(x)\right]^{\frac{1}{q}} \quad = \left[\int_X |f(x)|^{p'\cdot q'} d\mu(x)\right]^{\frac{1}{q'}}\end{aligned}$$

因此，我们有

$$\left[\int_X |f(x)|^p d\mu(x)\right]^{\frac{1}{p}} \leq \left[\int_X |f(x)|^{p\cdot q} d\mu(x)\right]^{\frac{1}{pq}} \quad \leq \left[\int_X |f(x)|^{p'\cdot q'} d\mu(x)\right]^{\frac{1}{p'q'}} \quad = \left[\int_X |f(x)|^{p'} d\mu(x)\right]^{\frac{1}{p'}}$$

从而证明了 $m(p)$ 是单调增加的。

现在假设对于 $1\leq p$，有 $m(p)<\infty$。由于 $m(p)$ 单调递增，$m(p)$ 在 $p\to\infty$ 时存在有限极限。为了找到这个极限，我们使用控制收敛定理(dominated convergence theorem)。令 $g(x)=|f(x)|^p$，对所有 $x\in X$，设其模长的上界为 M。那么对于所有 $x\in X$，有 $g(x)\leq M^p$，且 M^p 可积，因为 $\mu(X)=1$。因此，我们可以使用控制收敛定：

$$\lim_{p\to\infty} m(p) = \lim_{p\to\infty}\left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} = \left(\int_X \lim_{p\to\infty} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{\infty}} = \max |f(x)| : x\in X.$$

$\square$