(a) If $T_n \to T$ in norm, we have, for any vector $f \in \mathcal{H}$, that

$$\|T_n f - Tf\| = \|(T_n - T)f\| \leq \|(T_n - T)\|\|f\| \to 0$$

since $\|(T_n - T)\| \to 0$ and $\|f\|$ is finite.

If $T_n \to T$ in the sense of strong convergence, we have

$$(T_n f, g) - (Tf, g) = \big((T_n - T)f, g\big) \to 0$$

since $(T_n - T)f \to 0$ and inner product is a continuous operation.

(b) Let $\mathcal{H} = l^2$ consisting of number sequences that are square summable. The element $x_n = (0, .., 0, 1, 0, \dots)$ which has the only non-zero element 1 at the n-th slot. Let T be defined by doing inner product with x_n. For any $y = (y_1, y_2, \dots)$, we have

$$Ty = (x_n, y) = y_n \to 0$$

where $y_n \to 0$ is required by the "square summable" condition. Thus T_n converges in the strong sense to the operator T corresponding to doing an inner product with the zero vector (which is simply the zero/ null operator 0)

However, T_n doesn't converge strongly to 0 because each T_n has norm $\|T_n\| = 1$

Still, let $\mathcal{H} = l^2$ and denote T_n by a translation to the right by n, which maps a vector $x = (x_1, x_2, \dots)$ to

$$T_n x = (0, \dots, 0, x_1, x_2, \dots)$$

where there are n-0s in front of x_1. Notice that

$$(T_n x, y) = x_1 b_{n+1} + x_2 b_{n+2} + \cdots$$

Using Cauchy-Schwarz to get

$$|(T_n x, y)| \leq \|x\|\|(0, 0, \dots, 0, b_{n+1}, b_{n+2})\|$$

and $\|(0, 0, \dots, 0, b_{n+1}, b_{n+2})\| \to 0$, again due to the "square summable" condition. Thus T_n converges to the zero operator in the weak sense. However, obviously it is not in the strong sense.

(c) **Are you sure there is no mistake on the assumption? Finite rank operators are dense in the space of compact operators, but I never heard anyone mentioning it is possible to extend this to all bounded operators.**