(a) Since ψ_1, ψ_3 are normalized state, we have

$$\int \Psi^*(x)\Psi(x)dx = |A|^2 \int (4\psi_1^*(x)\psi_1(x) - 2i\psi_1^*(x)\psi_3(x) + 2i\psi_3^*(x)\psi_1(x) + \psi_3^*(x)\psi_3(x))dx$$

$$= |A|^2(4+1) = 5|A|^2 = 1$$

where we have used the fact that different eigenstates are orthogonal to cancel out the two cross terms in the middle, i.e.,

$$\int \psi_1^*(x)\psi_3(x)dx = 0$$

as well as the normalization condition to make

$$\int \psi_1^*(x)\psi_1(x)dx = \int \psi_3^*(x)\psi_3(x)dx = 1$$

Thus, we have

$$A = \frac{1}{\sqrt{5}}e^{i\theta}$$

where θ will give a global phase but for convenience we usually take $\theta = 0$ and so $A = \frac{1}{\sqrt{5}}$

(b) As mentioned in the hint, the condition

$$\psi_n(x) = \psi_n(a-x)$$

indicates that the wave function is symmetric about the center of the well. By symmetry, we know that the probability of occurrence of the particle in the left half of the well should equal that in the right half. Rigorously, we have, for odd m, n, that

$$\int_0^{a/2} \psi_m^*(x)\psi_n(x)\,dx = \int_{a/2}^{a} \psi_m^*(x)\psi_n(x)\,dx = \frac{1}{2}\int_0^a \psi_m^*(x)\psi_n(x)\,dx$$

Thus, when we calculate

$$\int_0^{a/2} \Psi^*(x)\Psi(x)\,dx$$

we always get one half from the total integral

$$\int_0^a \Psi^*(x)\Psi(x)\,dx = 1$$

Thus, the probability is simply 1/2.

(c) This energy corresponds to the $n = 2$ state, which unfortunately does not exist in our wavefunction. Thus, the probability of getting this energy at $t = 0$ is directly 0.

(d) The time-dependent solution to the Schrödinger equation will introduce the factor

$$e^{-i\omega_n t}$$

with

$$\hbar\omega_n = E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

to the time-independent solution $\psi_n(x)$. However, this time factor will be a factor 1 when we do the inner product for the time dependent solution

$$\int \psi_n^*(x,t)\psi_n(x,t)dx = e^{-i\omega_n t}e^{i\omega_n t}\int \psi_n^*(x)\psi_n(x)dx = \int \psi_n^*(x)\psi_n(x)dx$$

While for different eigenstates, the orthogonality condition always hold so they will cancel each other out. Thus, we know that the result should still be 1/2.