(a). If $f(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}$ and $g(x)=\sum_{n=-\infty}^{\infty} b_n e^{inx}$ are two functions in H, then
$$|Tf(x)| = \left|\sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx}\right| \quad \leq \sum_{n=-\infty}^{\infty} |\lambda_n||a_n| \ \leq \left(\sup_n |\lambda_n|\right) \sum_{n=-\infty}^{\infty} |a_n| \quad = \left(\sup_n |\lambda_n|\right) |f|.$$
Similarly, we have $|Tg(x)| \leq (\sup_n |\lambda_n|)|g|$. Thus, we have
$$|Tf|^2 = \int_{-\pi}^{\pi} |Tf(x)|^2 dx \quad \leq \int_{-\pi}^{\pi} \left(\sup_n |\lambda_n|\right)^2 |f(x)|^2 dx \ = \left(\sup_n |\lambda_n|\right)^2 |f|^2.$$
Taking the square root of both sides, we obtain $|Tf| \leq \sup_n |\lambda_n||f|$, which shows that T is a bounded operator on H, and that $|T| = \sup_n |\lambda_n|$.

(b). $f \in H$ and T is a bounded operator on H. In other words, we want to show that for any $f \in H$ and $h \in \mathbb{R}$, $\sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx} = \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{in(x-h)}$, where $f(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is the Fourier series of f. To see this, we have
$$T(\tau_h(f))(x) = T(f(x-h)) = \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{in(x-h)} \ \tau_h(T(f))(x) \quad = T(f(x-h)) = \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{in(x-h)},$$
where the last equality follows from the fact that T commutes with translations. Therefore, we have shown that T commutes with translations.

(c). Suppose that T is a bounded operator on H that commutes with translations. We want to show that T is a Fourier multiplier operator, i.e., there exists a bounded sequence such that for any $f \in H$ with Fourier series$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}$, we have $T(f)(x)=\sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx}$. To do this, we consider the functions $T(e^{inx})$ for each $n \in \mathbb{Z}$. Since T commutes with translations, we have $T(e^{i(x+h)n}) = \tau_h(T(e^{inx}))$ for any $h \in \mathbb{R}$ and $n \in \mathbb{Z}$. Thus, we have
$\sum_{m=-\infty}^{\infty} \lambda_m e^{im(x+h)n} = \sum_{m=-\infty}^{\infty} \lambda_m e^{imxn} e^{-imh}$. Dividing both sides by e^{inx} and summing over$n \in \mathbb{Z}$, we get $\sum_{m=-\infty}^{\infty} \lambda_m e^{imh} = \sum_{m=-\infty}^{\infty} \lambda_m e^{imx}$, which implies that the sequence $\lambda_{n_{n=-\infty}}^{\infty}$ is Fourier coefficients of some function $g \in L^1([-\pi,\pi])$, i.e., $\lambda_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\mathrm{d}x$. Moreover, for any $f \in H$ with Fourier series $f(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}$, we have
$$T(f)(x) = \sum_{n=-\infty}^{\infty} a_n T(e^{inx})(x) = \sum_{n=-\infty}^{\infty} a_n \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt\right) e^{inx} \quad = \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx},$$ which shows that T is a Fourier multiplier operator with multiplier sequence $\lambda_{n_{n=-\infty}}^{\infty}$.
Therefore, we have shown that any bounded operator on H that commutes with translations is a Fourier multiplier operator.