(a)

$$\mathrm{cov}(\tilde{B}_t, \tilde{B}_s)$$

$$= \mathrm{cov}\left(\frac{1}{\sqrt{c}} B_{tc}, \frac{1}{\sqrt{c}} B_{ts}\right)$$

$$= \frac{1}{c}\,\mathrm{cov}(B_{tc}, B_{ts})$$

$$= \frac{1}{c} \cdot tc$$

$$= t$$

Thus

$\tilde{B}_t$ is a Brownian motion

(b)

Let $0 \geq t \leq s$. Then $0 \leq \frac{1}{s} \leq \frac{1}{t}$

$$\operatorname{cov}(\hat{B}_t, \hat{B}_s)$$

$$= \operatorname{cov}(t B_{\frac{1}{t}}, s B_{\frac{1}{s}})$$

$$= ts \operatorname{cov}(B_{\frac{1}{t}}, B_{\frac{1}{s}})$$

$$= ts \cdot \frac{1}{s}$$

$$= t.$$

Thus $\hat{B}_t$ is a Brownian motion

(C)

Let $t \le s$

$\operatorname{cov}(-B_t, -B_s)$

$= \operatorname{cov}(B_t, B_s)$

$= t$

Thus

$(-B_t)_{t \ge 0}$ is a Brownian motion

(d)

$$E\left(B_t^2 - t \mid B_{t-1}^2 - (t-1)\right)$$

$$= E\left(B_t^2 - t \mid B_{t-1}\right)$$

$$= E\left((B_t - B_{t-1} + B_{t-1})^2 - t \mid B_{t-1}\right)$$

$$= E\left((B_t - B_{t-1})^2 + 2B_{t-1}(B_t - B_{t-1}) + B_{t-1}^2 - t \mid B_{t-1}\right)$$

$$= E\left((B_t - B_{t-1})^2 \mid B_{t-1}\right) + 2B_{t-1}\,E(B_t - B_{t-1})$$
$$+ B_{t-1}^2 - t$$

$$= E(B_t - B_{t-1})^2 + 0 + B_{t-1}^2 - t$$

$$= 1 + B_{t-1}^2 - t$$

$$= B_{t-1}^2 - (t-1)$$

Thus

$B_t^2 - t$ is a martingale