1. (a, i.)

Using Levi-Civita notation, we have

$$\frac{d}{dt}\langle L_i\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{L_i}]\rangle$$
$$= \frac{i}{\hbar}\langle[\hat{H},\epsilon_{ijk}\hat{x_j}\hat{p_k}]\rangle$$
$$= \frac{i}{\hbar}\epsilon_{ijk}\langle[\hat{H},\hat{x_j}]\hat{p_k} + \hat{x_j}[\hat{H},\hat{p_k}]\rangle$$

where we have used the commutator relation

$$[A,BC] = [A,B]C + B[A,C]$$

Using the given commutator relation of $\hat{H}$ with $\hat{x}$ and $\hat{p}$, we have

$$\frac{i}{\hbar}\epsilon_{ijk}\langle[\hat{H},\hat{x_j}]\hat{p_k} + \hat{x_j}[\hat{H},\hat{p_k}]\rangle = \frac{i}{\hbar}\epsilon_{ijk}\left\langle -\frac{i\hbar}{m}\hat{p_j}\hat{p_k} + i\hbar\hat{x_j}\frac{\partial V}{\partial x_k}\right\rangle$$
$$= \left\langle\left(\frac{1}{m}\hat{p}\times\hat{p}\right)_i\right\rangle - \langle(\hat{x}\times\nabla V)_i\rangle$$
$$= -\langle(\hat{x}\times\nabla V)_i\rangle$$

where we have used the fact $\hat{p}\times\hat{p} = 0$

(a. ii.)

Because if Vonly depends on r, ∇V will only have a radial component, and so $\hat{x}\times\nabla V$ is proportional to $\hat{x}\times\hat{x} = 0$.

(b, i.)

Since the radial part and the spherical part are independent and the radial part is already normalized, we simply need to normal the spherical part. Using orthogonality, we have

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

(b. ii.)

For L^2, Y_{lm} is an eigenvector with eigenvalues $l(l+1)\hbar^2$. Thus with probability $|a|^2$, the result is 0 and with probability $|b|^2 + |c|^2 + |d|^2$, the result is $2\hbar^2$

(b, iii.)

For L_3, , Y_{lm} is an eigenvector with eigenvalues $m\hbar$. Thus, with probability $|a|^2 + |c|^2$, the result is 0, with probability $|b|^2$ the result is $\hbar$ and with probability $|d|^2$ the result is $-\hbar$

(b, iv.)

I honestly do not know what the problem means by "measured at the same time". Is it asking a theoretical possibility or experimental implementation? One thing one can say for sure is that L^2 commute with L_3, and so it is possible to measure both quantities to arbitrarily precision at the same time.

(c. i.)

L^2 will only care about the value l of a state u_{nlm}. Thus, with probability $|\alpha|^2 + |\beta|^2$ the result will be 0, and with probability $|\gamma|^2 + |\delta|^2 + |\epsilon|^2$ the result will be $2\hbar^2$

(c. ii)

Similar as what we have done before, L_3 only cares about m for a state u_{nlm}. Thus for a probability $|\alpha^2| + |\beta|^2 + |\delta|^2$ the result is 0, with probability $|\gamma|^2$ the result is $\hbar$ and with probability $|\epsilon|^2$ the result is $-\hbar$

(c. iii.)

Since the energy eigenvalues are E_{nl} depending on the first two indices of u_{nlm}. We see that with probability $|\alpha|^2$ the energy is E_{00}, with probability $|\beta|^2$ the energy is E_{10}, and with probability $|\gamma|^2 + |\delta|^2 + |\epsilon|^2$ the energy is E_{11}

(c. iv.)

The state will be $\alpha u_{000} + \beta u_{100} + \delta u_{110}$. After normalization, the result is

$$\frac{1}{\sqrt{|\alpha^2| + |\beta|^2 + |\delta|^2}} (\alpha u_{000} + \beta u_{100} + \delta u_{110})$$