(a)

Define

$$Y^*_{\cdot} = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1J_1} \\ Y_{21} \\ \vdots \\ Y_{2J_2} \\ \vdots \\ Y_{IJ_I} \end{bmatrix} \qquad \varepsilon^* = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1J_1} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{2J_2} \\ \vdots \\ \varepsilon_{IJ_I} \end{bmatrix}$$

$$X_{\cdot}^{(j)} = \begin{cases} 1 & \text{if the subject belongs to the } j\text{-th group} \\ 0 & \text{otherwise} \end{cases}$$

then

$$Y^*_i = \mu + \alpha_1 X_i^{(1)} + \alpha_2 X_i^{(2)} + \cdots + \alpha_I X_i^{(I)} + \varepsilon^*_i$$

$$i = 1, 2, \ldots \sum_{i=1}^{I} J_i .$$

$X^{(1)}, X^{(2)}, \ldots X^{(I)}$ are the predictors

and $\alpha_1, \alpha_2, \ldots \alpha_I$ are the coefficient

μ is the intercept

(b)

Write $X = \begin{bmatrix} 1 & & \\ \vdots & & \\ 1 & & \\ & 1 & \\ & \vdots & \\ & 1 & \\ & & 1 \\ & & \vdots \\ & & 1 \end{bmatrix}$

where for j-th column, we have J_j's 1.

Thus

$$Y^* = X\alpha + \varepsilon^*,$$

where $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_I \end{bmatrix}$

Thus

$$\hat{\alpha} = (X^T X)^{-1} X^T y$$

Note that $X^T X = \text{diag}(J_1, J_2, \ldots, J_I)$

Thus $(X^T X)^{-1} = \text{diag}(\frac{1}{J_1}, \frac{1}{J_2}, \ldots, \frac{1}{J_I})$

$$X^T y = \begin{pmatrix} \sum_{j=1}^{J_1} Y_{1j} \\ \sum_{j=1}^{J_2} Y_{2j} \\ \vdots \\ \sum_{j=1}^{J_I} Y_{Ij} \end{pmatrix}$$

Thus

$$\hat{\alpha}_i = \frac{1}{J_i} \sum_{j=1}^{J_i} Y_{ij} ,$$

(c) Let $N = \sum_{j=1}^{I} J_j$

The estimation of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{N-I} \sum_j (\hat{e}^*_j)^2 .$$

$$= \frac{1}{N-I} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \hat{e}_{ij}^2$$

$$= \frac{1}{N-I} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \left(Y_{ij} - \frac{1}{J_i} \sum_{j=1}^{J_i} Y_{ij} \right)^2$$