We will prove these two quantities are equal to $\coth(\pi a)$ directly.

For the right-hand side, it can be written into geometric series

$$1+2\sum_{n=0}^{\infty} e^{-2\pi a n} = 1+\frac{2e^{-2\pi a}}{1-e^{-2\pi a}}$$

which is now exactly $\coth(\pi a)$ since

$$\coth(\pi a) = \frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}} = \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} = 1+\frac{2e^{-2\pi a}}{1-e^{-2\pi a}}$$

Now we proceed to prove that the left-hand side is also $\coth(\pi a)$. We consider the Fourier expansion of $\coth(ax)$ in the interval $[-\pi, \pi]$. The b_n terms are all zero since the function is even. While for a_n we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh ax \, \mathrm{d}x = \frac{2\sinh \pi a}{\pi a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cosh ax \cos nx \, \mathrm{d}x = \frac{2[a\sinh \pi a \cos \pi n + \cancelto{0}{n\cosh \pi a \sin \pi n}]}{\pi(a^2+n^2)}$$

where the crossed-out term is due to $\sin(\pi n) = 0$. If you find the second integral hard to evaluate, first write out the exponential form

$$\cosh(ax)\cos(nx) = \frac{1}{2}\cos(nx)\,(e^{-ax}+e^{ax})$$

and you **should** know how to integrate $\cos(nx)\,e^{-ax}$ (integration by parts). If you don't know, go review first-year calculus.

We thus have the expansion

$$\cosh ax = \frac{\sinh \pi a}{\pi a} + \frac{2a\sinh \pi a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2+a^2}$$

Plug in $x = \pi$ to get

$$\coth(\pi a) = \frac{1}{\pi a} + \frac{1}{\pi a}\sum_{n=1}^{\infty}\frac{2}{a^2+n^2}$$

which is exactly

$$\coth(\pi a) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2}.$$

As for the formulae that the problem doesn't need you to prove. It makes use of the properties of meromorphic functions, or by doing analytical continuation as hinted by the question.