Q1.

a)

The likelihood function is

$$L = \frac{1}{\theta^n} \prod_{i=1}^{n} I(0 \le X_i \le \theta)$$

$$= \frac{1}{\theta^n} I(0 \le \min(X_i), \max(X_i) \le \theta)$$

$\frac{1}{\theta^n}$ is a decreasing function of θ.

So

$$\hat{\theta}_{MLE} = \max_{1 \le i \le n} X_i$$

b)

$$f(X; \theta) = \frac{1}{\theta^n}$$

$$\frac{\partial \log f(X, \theta)}{\partial \theta} = -\frac{n}{\theta}$$

So

$$I(\theta) = E\left(\frac{n}{\theta}\right)^2 = \frac{n^2}{\theta^2}$$

CR bound is $\frac{1}{I(\theta)} = \frac{\theta^2}{n^2}$.

c) Let $X_{max} = \max_{1 \le i \le n} X_i$

Then

$$p(X_{max} \le x) = p(X_i < x,\ i = 1, 2, \ldots, n)$$

$$= \prod_{i=1}^{n} p(X_i < x)$$

$$= \begin{cases} 1 & x > \theta \\ \left(\frac{x}{\theta}\right)^n & 0 \le x \le \theta \\ 0 & x \le 0 \end{cases}$$

So the density of X_{max} is

$$f_{max}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

$$E X_{max} = \int_0^\theta x \cdot \frac{n}{\theta^n} x^{n-1} \, dx$$

$$= \frac{n}{n+1} \theta$$

So

$$E\left(\frac{n+1}{n} X_{max}\right) = \theta$$

$\frac{n+1}{n} X_{max}$ is an unbiased estimator of θ

$$E\left(\frac{n+1}{n}X_{max}\right)^2$$

$$=\left(\frac{n+1}{n}\right)^2 E X_{max}^2$$

$$=\left(\frac{n+1}{n}\right)^2 \int_0^\theta x^2 \cdot \frac{n}{\theta^n} x^{n-1} \, d\theta$$

$$=\left(\frac{n+1}{n}\right)^2 \frac{n}{\theta^n} \int_0^\theta x^{n+1} \, d\theta$$

$$=\frac{(n+1)^2}{n} \frac{1}{\theta^n} \left.\frac{x^{n+2}}{n+2}\right|_0^\theta$$

$$=\frac{(n+1)^2}{n} \frac{1}{\theta^n} \frac{\theta^{n+2}}{n+2}$$

$$=\frac{(n+1)^2}{n(n+2)} \theta^2$$

$$\mathrm{var}\left(\frac{n+1}{n}X_{max}\right)$$

$$=E\left(\frac{n+1}{n}X_{max}\right)^2-\theta^2$$

$$=\frac{(n+1)^2}{n(n+2)}\theta^2-\theta^2$$

$$=\frac{1}{n(n+2)}\theta^2$$

This is less then the CR lower bound.

The reason is that the support of X depends on θ. So X is not regular.

d.

$$E(2\bar{X}_n) = 2 E\bar{X}_n = \frac{2}{n}\sum_{i=1}^{n} E X_i$$

$$= \frac{2}{n} \cdot n \cdot \frac{\theta}{2}$$

$$= \theta$$

So $2\bar{X}_n$ is also unbasied

$$var(2\bar{X}_n) = 4 \, var \bar{X}_n = \frac{4}{n} var X_i$$

$$var X_i = E X_i^2 - (E X_i)^2$$

$$= \int_0^\theta x^2 \cdot \frac{1}{\theta} dx - \left(\frac{\theta}{2}\right)^2$$

$$= \frac{1}{3}\theta^2 - \frac{1}{4}\theta^2$$

$$= \frac{1}{12}\theta^2.$$

So

$$\mathrm{var}(2\bar{X}_n) = \frac{4}{n}\frac{1}{12}\theta^2 = \frac{1}{3n}\theta^2.$$

This is larger than the CR bound and the variance of MLE in c).

e.

$2\bar{X}_n$ is not the UMVUE

$\frac{n+1}{n} X_{max}$ is the UMVUE

Note that

$$f(x;\theta) = \frac{1}{\theta^n} I(0 \le x_{min}) I(X_{max} \le \theta)$$

$$= I(0 \le X_{min}) \frac{1}{\theta^n} I(X_{max} \le \theta)$$

So X_{max} is a sufficient statistic by using the factorization theorem

So by Lehmann-Scheffe's theorem,

$\frac{n+1}{n} X_{max}$ is the UMVUE,

2.

a)

The joint density of $(X_1 \ldots X_n)$ is

$$f(x_1 \ldots x_n; \theta) = \beta^{-(\alpha+1)n} \left(\Gamma(\alpha+1)\right)^{-n} \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha}$$

$$\exp\left(-\frac{1}{\beta}\sum_{i=1}^{n} x_i\right)$$

$$\prod_{i=1}^{n} I_{[0,\infty)}(x_i)$$

Thus by the factorization theorem

$\left(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right)$ are the joint sufficient statistic for (α, β)

b)

$f(x_1, \dots, x_n; \theta)$

$$= \beta^{-(\alpha+1)n} (\Gamma(\alpha+1))^{-n} \left(\prod_{i=1}^{n} x_i\right)^{\alpha} \exp\left(-\frac{1}{\beta}\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} I(0 \le x_i < \infty)$$

$$= \beta^{-(\alpha+1)n} (\Gamma(\alpha+1))^{-n} \exp\left(\alpha \log \prod_{i=1}^{n} x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} I(0 \le x_i < \infty)$$

This is a exponential family

So $\left(\prod_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i\right)$ is the complete statistic.

Thus $\left(\prod_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i\right)$ is sufficient and complete.

Therefore, it is minimally sufficient.

3.

a.

$$f(x_1 \ldots x_n | \theta) = \theta^n \cdot (1-\theta)^{\sum_{i=1}^{n} x_i}$$

$$\log f(x_1 \ldots x_n | \theta) = n \log \theta + \sum_{i=1}^{n} x_i \log(1-\theta)$$

$$\frac{\partial \log f(x_1 \ldots x_n | \theta)}{\partial \theta} = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta}$$

$$\frac{\partial^2 \log f(x_1 \ldots x_n | \theta)}{\partial \theta^2} = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} x_i}{(1-\theta)^2}$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f(x_1 \ldots x_n | \theta)}{\partial \theta^2}\right)$$

$$= E\left(\frac{n}{\theta^2} + \frac{\sum_{i=1}^{n} x_i}{(1-\theta)^2}\right)$$

$$= \frac{n}{\theta^2} + \frac{nE x_i}{(1-\theta)^2}$$

$$E X_i = \sum_{i=0}^{\infty} \theta (1-\theta)^{i} i$$

$$= \theta \sum_{i=0}^{\infty} (1-\theta)^{i} i$$

$$= \theta(1-\theta) \sum_{i=0}^{\infty} (1-\theta)^{i-1} i$$

$$= \theta(1-\theta) \left[\frac{d}{d\theta} \left(-\sum_{i=0}^{\infty} (1-\theta)^{i} \right) \right]$$

$$= \theta(1-\theta) \frac{d}{d\theta} \left(-\frac{1}{\theta} \right)$$

$$= \theta(1-\theta) \frac{1}{\theta^2}$$

$$= \frac{1-\theta}{\theta}$$

So

$$I(\theta) = \frac{n}{\theta^2} + \frac{n \cdot \frac{1-\theta}{\theta}}{(1-\theta)^2}$$

$$= \frac{n}{\theta^2} + \frac{n}{\theta(1-\theta)} = \frac{n}{\theta}\left(\frac{1}{\theta} + \frac{1}{1-\theta} \right)$$

$$= \frac{n}{\theta^2(1-\theta)}$$

So

CR lower bound for θ is $\frac{\theta^2(1-\theta)}{n}$

and

CR lower bound for $1-\theta$ is $\frac{(1-\theta)^2\theta}{n}$

b. By the factorization theorem,

$\overline{X}_n = \frac{1}{n}\sum_{i=1} X_i$ is an sufficient

statistic.

$$E\,\overline{X}_n = \frac{1-\theta}{\theta}$$

So by Lehmann-scheffe's theorem,

$\overline{X}_n$ is the UMVUE for $\frac{1-\theta}{\theta}$

4.

a)

The likelihood function is

$$L(\theta) = \theta^n \cdot \left(\prod_{i=1}^{n} X_i\right)^{\theta-1}$$

$$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log X_i .$$

$$\frac{\partial \log L(\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(X_i) = 0$$

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n}\log(X_i)} = -\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\log X_i}$$

The MLE of

$\frac{\theta}{\theta+1}$ is $\frac{\hat{\theta}_{MLE}}{1+\hat{\theta}_{MLE}}$.

b)

$\sum_{i=1}^{n} X_i$ is not sufficient for θ.

By the factorization theorem.

$\prod_{i=1}^{n} X_i$ or $\sum_{i=1}^{n} \log X_i$

is sufficient.

$$f(x_1 .. x_n) = \theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta-1} I(0 < X_i < 1, i = 1, 2 .. n)$$

$$= \theta^n \exp\left((\theta-1) \sum_{i=1}^{n} \log(X_i)\right) I(0 < X_i < 1, i = 1, .. n)$$

So this is an exponental family

Thus $\sum_{i=1}^{n} \log X_i$ is a complete statistic

a) Define $Y = \log X$.

we first derive the density of Y.

$$f_Y(y) = f_X(e^y) \cdot \left| \frac{dx}{dy} \right|$$

$$= \theta e^{(\theta-1)y} e^y$$

$$= \theta e^{\theta y} \qquad y \le 0$$

$$EY = \int_{-\infty}^{0} \theta e^{\theta y} \cdot y \, dy$$

$$= \int_{-\infty}^{0} y \, d e^{\theta y}$$

$$= y \cdot e^{\theta y} \Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{\theta y} dy$$

$$= - \frac{e^{\theta y}}{\theta} \Big|_{-\infty}^{0}$$

$$= - \frac{1}{\theta}$$

Thus by Lehmann-Scheffe's theorem,

$$- \frac{1}{n} \sum_{i=1}^{n} Y_i = - \frac{1}{n} \sum_{i=1}^{n} \log X_i$$

is the UMVUE of $\frac{1}{\theta}$.

Let $Z_i = -Y_i$

Then $Z_i \sim \exp(\theta)$

$$-\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} Z_i \sim \text{Gamma}(n, \theta)$$

$$E \frac{1}{-\frac{1}{n}\sum_{i=1}^{n} Y_i} = n E \frac{1}{\sum_{i=1}^{n} Z_i}$$

$$= n \int_0^{\infty} \frac{1}{x} \frac{1}{\Gamma(n)} \theta^n \cdot x^{n-1} e^{-\theta x} dx$$

$$= n \int_0^{\infty} \frac{1}{\Gamma(n)} \theta^n x^{n-2} e^{-\theta x} dx$$

$$= n \frac{\theta \Gamma(n-1)}{\Gamma(n)} \int_0^{\infty} \frac{1}{\Gamma(n-1)} \theta^{n-1} x^{n-2} e^{-\theta x} dx$$

$$= \theta n \cdot \frac{1}{n-1} = \frac{n}{n-1} \theta$$

So by Lehmann-Scheffe's theorem,

$$-\frac{n-1}{n} \frac{1}{\bar{Y}} = -\frac{n-1}{n} \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \log X_i}$$

is the UMVUE of θ

5.

a).

The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-n\lambda}}{\prod_{i=1}^{n} x_i!} \lambda^{\sum_{i=1}^{n} x_i}$$

$$\log L(\lambda) = -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - \log \prod_{i=1}^{n} x_i!$$

$$\frac{\partial \log L(\lambda)}{\partial \lambda} = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i$$

So $\hat{\lambda}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$

MLE for $g(\lambda)$ is $e^{-\bar{x}}(1+\bar{x})$.

b)

Note that $T(X) = \sum_{i=1}^{n} X_i$ is the sufficient and complete statistic.

Moreover, let $W_0 = I(X_1 = 0)$

$W_1 = I(X_1 = 1)$

Then

$$EW_0 = e^{-\lambda}$$

$$EW_1 = \lambda e^{-\lambda}$$

So by Lehmann-Scheffe's,

$E(W_0 | T)$ is the UMVUE of $e^{-\lambda}$

$E(W_1 | T)$ is the UMVUE of $\lambda e^{-\lambda}$.

$$E(W_0 | T) = P(X_1 = 0 | T)$$

$$= \frac{P(X_1 = 0, T = t)}{P(T = t)}$$

$$= \frac{P(X_1 = 0)\, P(X_2 + \cdots + X_n = t)}{P(X_1 + \cdots + X_n = t)}$$

$$= \frac{e^{-\lambda} \cdot e^{-(n-1)\lambda} ((n-1)\lambda)^t / t!}{e^{-(n\lambda)} (n\lambda)^t / t!}$$

$$= \frac{(n-1)^t}{n^t}.$$

$$E(W_1 \mid T) = \frac{P(X_1 = 1)\, P(X_2 + \cdots + X_n = t-1)}{P(X_1 + \cdots + X_n = t)}$$

$$= \frac{e^{-\lambda} \lambda \cdot e^{-(n-1)\lambda} \cdot ((n-1)\lambda)^{t-1} / (t-1)!}{e^{-n\lambda} (n\lambda)^t / t!}$$

$$= t \cdot \frac{(n-1)^{t-1}}{n^t}$$

So

$$\frac{(n-1)^T}{n^T} + T \cdot \frac{(n-1)^{T-1}}{n^T}$$ is the UMVUE

of $e^{-\lambda}(\lambda+1)$