1.

$1+z+z^2+z^3 = (1+z)(1+z^2)$.

So it has 3 roots $z=-1$, $z=i$, $z=-i$

Since $p(z^4) + z\,q(z^4) + z^2 r(z^4)$

is divisible by $1+z+z^2+z^3$.

we have

$$\begin{cases} p(1) - q(1) + r(1) = 0 \\ p(1) + i\,q(1) - r(1) = 0 \\ p(1) - i\,q(1) - r(1) = 0 \end{cases}$$

So $p(1) = r(1) = q(1) = 0$

Thus $p(z)$ is divisble by $z-1$

2.

$$(1+z)^5 = 1+i$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \qquad e^{\pi i}$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 2^{\frac{1}{2}} e^{\frac{\pi}{4}i}$$

$$= 2^{\frac{1}{2}} e^{\left(\frac{\pi}{4}+2n\pi\right)i}$$

Thus

$$1+z = 2^{\frac{1}{10}} e^{\frac{1}{5}\left(\frac{\pi}{4}+2n\pi\right)i}$$

$$= 2^{\frac{1}{10}} e^{\frac{1}{20}(\pi+8n\pi)i}$$

$$z = 2^{\frac{1}{10}} e^{\frac{1}{20}(\pi+8n\pi)i} - 1$$

$$z_1 = 2^{\frac{1}{10}} e^{\frac{i\pi}{20}} - 1$$

$$z_2 = 2^{\frac{1}{10}} e^{\frac{9i\pi}{20}} - 1$$

$$z_3 = 2^{\frac{1}{10}} e^{\frac{17i\pi}{20}} - 1$$

$$z_4 = 2^{\frac{1}{10}} e^{\frac{25i\pi}{20}} - 1$$

$$z_5 = 2^{\frac{1}{10}} e^{\frac{33}{20}i\pi} - 1$$

3.

Let $z \in D$

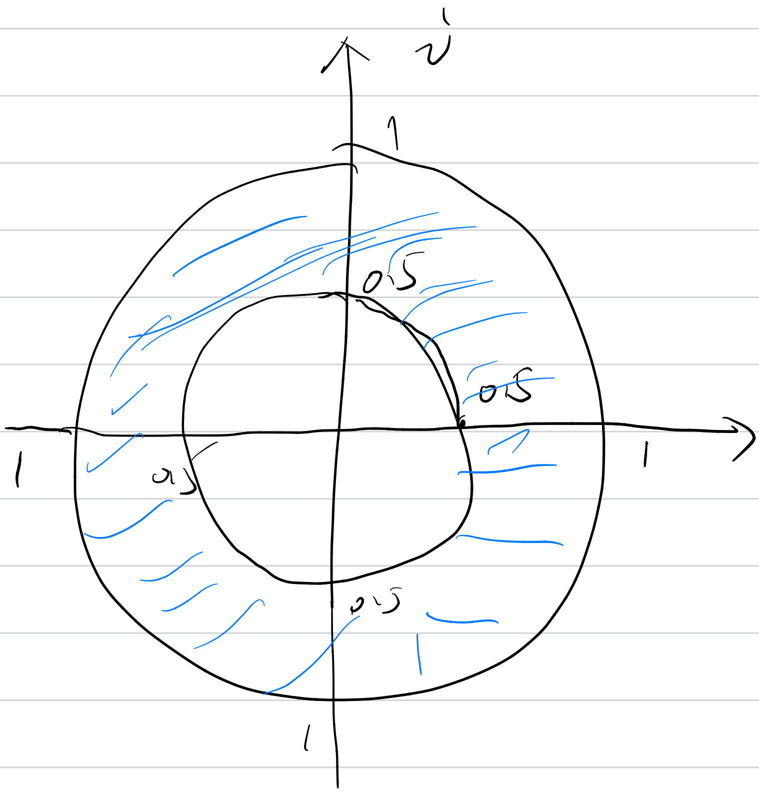
Then $z = a + bi$ with $0.5 \leq \sqrt{a^2+b^2} \leq 1$

$$f(z) = iz = -b + ai$$

$$|f(z)| = \sqrt{a^2+b^2} = |z|$$

So

$$f(D) = D$$



4.

(a)

Let X denotes the number of heads

Then $X \sim B(6, 0.5)$

$P(X \geq 4)$

$= P(X=4) + P(X=5) + P(X=6)$

$= \binom{6}{4} 0.5^4 0.5^2 + \binom{6}{5} 0.5^5 0.5 + \binom{6}{6} 0.5^6$

$= 0.34375$

(b)

At least two tails $\Leftrightarrow$ $X \leq 4$

The number of heads is more than the number of tails

$\Leftrightarrow$ $X \geq 4$.

So $p(X \le 4 \mid X \ge 4) = \dfrac{p(X \le 4, X \ge 4)}{p(X \ge 4)}$

$= \dfrac{p(X = 4)}{p(X \ge 4)}$

$= \dfrac{\binom{6}{4} 0.5^4 0.5^2}{0.34375}$

$= 0.6818$.

5.

(a)

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= 0.45 + 0.25 - 0.5$$

$$= 0.2$$

(b)

$$P(A' \cap B') = P((A \cup B)')$$

$$= 1 - P(A \cup B)$$

$$= 1 - 0.5$$

$$= 0.5$$

(c)

$$P(A' \mid B)$$

$$= \frac{P(A' \cap B)}{P(B)} = \frac{P(A') - P(A' \cap B')}{P(B)}$$

$$= \frac{0.55 - 0.5}{0.25} = \frac{0.05}{0.25} = 0.2$$

6.

(a)

Let $Y=1$ denote the car has a defect and $X=1,2,3$ to denote the car is produng from the three plants respectivly

$$P(Y=1) = P(Y=1 \mid X=1)\,P(X=1)$$
$$+P(Y=1 \mid X=2)\,P(X=2)$$
$$+P(Y=1 \mid X=3)\,P(X=3)$$

$$= 0.02 \times 0.5 + 0.05 \times 0.35 + 0.01 \times 0.15$$

$$= 0.029$$

(b)

$$p(X=2 \mid Y=1)$$

$$= \frac{p(X=2, Y=1)}{p(Y=1)}$$

$$= \frac{p(Y=1 \mid X=2)\, p(X=2)}{p(Y=1)}$$

$$= \frac{0.05 \times 0.35}{0.029} = 0.6034$$

7.

(a) Since X can only take $0, 1$

X^2 can only take $0, 1$

Thus all the possible values of Y is

$0, 1, 2 \ldots n.$

(b) note that $X_i^2 = X_i$

Thus

$$Y = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i \sim B(n, p)$$

$P(Y=r) = P($ for $X_1 \ldots X_n$, we have 1 with r times and 0 with $n-r$ times$)$

$$= \binom{n}{r} \cdot p^r \cdot (1-p)^{n-r}$$

c)

$$EY = E\sum_{i=1}^{n} X_i$$

$$= \sum_{i=1}^{n} EX_i$$

$$= n \cdot p$$

$$\text{var}Y = \text{var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= n\,\text{var}X_i$$

$$= np(1-p)$$