(1) $|\lambda E-A|=\begin{vmatrix}\lambda-2 & -1 & 0\\ -1 & \lambda-1 & 1\\ 0 & 1 & \lambda-2\end{vmatrix}=(\lambda-2)^2(\lambda-1)-(\lambda-2)-(\lambda-2)=(\lambda-2)^2(\lambda-3)=0 \Rightarrow \lambda_1=2,\ \lambda_2=2,\ \lambda_3=3.$

$\Rightarrow A\sim\begin{pmatrix}2&0&0\\0&2&0\\0&0&3\end{pmatrix}.$

So. $(\vec{x}^TAx)_{max}=12 \quad (\vec{x}Ax)_{min}=8$

$\vec{x}_{max}=(\frac{2\sqrt{3}}{3},\frac{2\sqrt{3}}{3},-\frac{2\sqrt{3}}{3})^T.$

$\vec{x}_{min}=(\sqrt{2},0,\sqrt{2})^T$

(2) (a) $A=\begin{pmatrix}2&-2&-4\\-2&5&-2\\-4&-2&2\end{pmatrix}$

(b) $|\lambda E-A|=\begin{vmatrix}\lambda-2&2&4\\2&\lambda-5&2\\4&2&\lambda-2\end{vmatrix}=(\lambda+3)(\lambda-6)^2=0.\ \lambda_1=6,\lambda_2=6,\ \lambda_3=-3$

$\Rightarrow A\sim\begin{pmatrix}6&0&0\\0&6&0\\0&0&-3\end{pmatrix}$

$6E-A=\begin{pmatrix}4&2&4\\2&1&2\\4&2&4\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2\\0&0&0\\0&0&0\end{pmatrix}.\quad (6E-A)X=0\Rightarrow\begin{cases}X_1=(1,-2,0)^T\\X_2=(1,0,-1)^T\end{cases}$

$-3E-A=\begin{pmatrix}-5&2&4\\2&-8&2\\4&2&-5\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&-1\\0&2&-1\\0&0&0\end{pmatrix}\quad (-3E-A)X=0\Rightarrow X_3=(2,1,2)$

$\alpha_1=X_1=(1,-2,0)^T$

$\alpha_2=X_2-\frac{(X_2,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1=(\frac{4}{5},\frac{2}{5},-\frac{5}{5})$

$\alpha_3=X_3-\frac{(X_3,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1-\frac{(X_3,\alpha_2)}{(\alpha_2,\alpha_2)}\alpha_2=(2,1,2)$

$\beta_1=\frac{\alpha_1}{\|\alpha_1\|}=(\frac{\sqrt{5}}{5},-\frac{2\sqrt{5}}{5},0)^T$

$\beta_2=\frac{\alpha_2}{\|\alpha_2\|}=(\frac{4\sqrt{5}}{15},\frac{2\sqrt{5}}{15},-\frac{\sqrt{5}}{3})^T$

$\beta_3=\frac{\alpha_3}{\|\alpha_3\|}=(\frac{2}{3},\frac{1}{3},\frac{2}{3})^T$

$P=(\beta_1,\beta_2,\beta_3).\qquad \underline{A=P^T\begin{pmatrix}6&0&0\\0&6&0\\0&0&-3\end{pmatrix}P}$