Q1:

(a)

$$EX = 0.15\times0.18 + 0.15\times(-0.05) + 0.1\times0.08 + 0.2\times0.1 + 0.25\times0.15 + 0.15\times(-0.2)$$

$$= 0.055$$

$$EY = 0.15\times0.08 + 0.15\times(-0.03) + 0.1\times0.1 + 0.2\times0.06 + 0.25\times(0.05) + 0.15\times0.02$$

$$= 0.02$$

$$EX^2 = 0.15\times0.18^2 + 0.15\times(-0.05)^2 + 0.1\times0.08^2 + 0.2\times0.1^2 + 0.25\times0.15^2 + 0.15\times(-0.2)^2$$

$$= 0.0195$$

$$\text{var } X = EX^2 - (EX)^2$$

$$= 0.016475$$

Similarly

$$EY^2 = 0.0035$$

$$\text{var } Y = EY^2 - (EY)^2 = 0.0031$$

$$\mathrm{cov}(X,Y) = EXY - EXEY$$

$$EXY = 0.15 \times (0.18 \times 0.08) + 0.15 \times (-0.05) \times (-0.03)$$

$$+ 0.1 \times 0.08 \times 0.1 + 0.2 \times 0.1 \times 0.06$$

$$+ 0.25 \times 0.15 \times (-0.05) + 0.15 \times (-0.2) \times 0.02$$

$$= 0.00191$$

$$\mathrm{cov}(X,Y) = 0.00191 - 0.055 \times 0.02$$

$$= 0.00081$$

(b)

$$L = 0.75X + 0.25Y$$

$$EL = 0.75EX + 0.25EY$$

$$= 0.04625$$

$$\operatorname{var} L = 0.75^2 \operatorname{var} X + 0.25^2 \operatorname{var} Y + 2 \times 0.75 \times 0.25 \times \operatorname{cov}(X,Y)$$

$$= 0.00976$$

c)

First, the minimum variance portfolio

$$\min_{w_X} \; w_X^2 \operatorname{var} X + (1-w_X)^2 \operatorname{var} Y + 2w_X(1-w_X)\operatorname{cov}(X,Y)$$

$$w_X = \frac{\operatorname{var} Y - \operatorname{cov}(X,Y)}{\operatorname{var} X + \operatorname{var} Y - 2\operatorname{cov}(X,Y)}$$

$$= 0.12754$$

$$w_Y = 1 - w_X = 0.87246$$

The optimal risky portfolio is given by

$$w_X = \frac{(EX - r_f)\,\mathrm{var}Y - (EY - r_f)\,\mathrm{cov}(X,Y)}{(EX - r_f)\,\mathrm{var}Y + (EY - r_f)\,\mathrm{var}X - (EX + EY - 2r_f)\,\mathrm{cov}(X,Y)}$$

$$= 1.3537$$

$$w_Y = 1 - 1.3537 = -0.3537$$

(d)

Let V denote the optimal risky portfolio

$$V = 1.3537X - 0.3537Y$$

$$EV = 0.0674$$

$$\mathrm{var}\, V = 0.0298$$

$$0.0674 \cdot w + 0.02(1-w) = 0.1$$

$$0.0474w = 0.08$$

$$w = 1.6878$$

So the weight for X, Y and risk free assets are

$$(1.6878 \times 1.3537,\ 1.6878 \times (-0.3537),\ -0.6878)$$

$$= (2.2848,\ -0.5970,\ -0.6878)$$

2.

(a)

$$q_1 = \frac{1}{3} \times 1 + \frac{2}{3} \times 1 = 1$$

$$q_2 = \frac{1}{3} \times 0 + \frac{2}{3} \times 1 = \frac{2}{3}$$

(b)

For Bob,

$$U_{Bob} = \ln(X_0) + \frac{1}{3}\ln(X_1) + \frac{2}{3}\ln(X_2)$$

With good state

$$U_{Bob}^{good} = \ln(X_0) + \frac{1}{3}\ln(3500 - X_0)$$

with bad state

$$U_{Bob}^{bad} = \ln(X_0) + \frac{2}{3}\ln(2500 - X_0)$$

Thus

$$U = \frac{1}{3} U_{Bob}^{good} + \frac{2}{3} U_{Bob}^{bad}$$

$$= \ln(X_0) + \frac{1}{9}\ln(3500 - X_0) + \frac{4}{9}\ln(2500 - X_0)$$

$$\frac{du}{dx_0} = 0$$

$$\frac{1}{x_0} - \frac{1}{9}\frac{1}{3500-x_0} - \frac{4}{9}\frac{1}{2500-x_0} = 0$$

$$9(3500-x_0)(2500-x_0) = x_0(2500-x_0) - 4x_0(3500-x_0)$$

$$x_0 = 1672.5$$

If state = good, $x_1 = 3500 - 1672.5 = 1827.5$

If state = bad $x_2 = 2500 - 1672.5 = 827.5$

For Ben.

$$U_{Ben}^{good} = \frac{5}{4}\ln(x_0) + \frac{1}{3}\ln(2500 - x_0)$$

$$U_{Ben}^{Bad} = \frac{5}{4}\ln(x_0) + \frac{2}{3}\ln(3500 - x_0)$$

$$U = \frac{1}{3}U_{Ben}^{good} + \frac{2}{3}U_{Ben}^{Bad}$$

$$= \frac{5}{4}\ln x_0 + \frac{1}{9}\ln(2500 - x_0) + \frac{4}{9}\ln(3500 - x_0)$$

$$\frac{du}{dx_0} = 0$$

$$\frac{5}{4}\frac{1}{x_0} - \frac{1}{9}\frac{1}{2500 - x_0} - \frac{4}{9}\frac{1}{3500 - x_0} = 0$$

$$45(2500 - x_0)(3500 - x_0) - 4x_0(3500 - x_0) - 16x_0(2500 - x_0) = 0$$

$$x_0 = 2100$$

if state = good $\quad x_1 = 2500 - 2100 = 400$

if state = bad $\quad x_2 = 3500 - 2100 = 1400$

c) Both of them should borrow from the market. Bob needs to borrow 1725

Bob needs to borrow 600.