1. $A=\frac{1}{5}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}$

(a) $|A-\lambda E|=\begin{vmatrix}\frac{1}{5}-\lambda&\frac{2}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{1}{5}-\lambda&\frac{2}{5}\\\frac{2}{5}&\frac{2}{5}&\frac{1}{5}-\lambda\end{vmatrix}$

$$=\left(\frac{1}{5}-\lambda\right)^3+\left(\frac{2}{5}\right)^3+\left(\frac{2}{5}\right)^3-\frac{2}{5}\times\left(\frac{1}{5}-\lambda\right)\times\frac{2}{5}-\frac{2}{5}\times\frac{2}{5}\times\left(\frac{1}{5}-\lambda\right)-\left(\frac{1}{5}-\lambda\right)\times\frac{2}{5}\times\frac{2}{5}$$

$$=-\lambda^3+\frac{3}{5}\lambda^2+\frac{9}{25}\lambda+\frac{1}{25}$$

$$=25\lambda^3-15\lambda^2-9\lambda-1$$

$$=(5\lambda+1)^2\cdot(\lambda-1)$$

$$=0$$

解得三个特征值分别为$\lambda_1=\lambda_2=-\frac{1}{5},\lambda_3=1$

下面求各个特征值对应的特征向量

当$\lambda_1=\lambda_2=-\frac{1}{5}$ 时，将$\lambda=-\frac{1}{5}$ 代入$(A-\lambda E)x=0$，即$\left(A+\frac{1}{5}E\right)x=0$，亦即

$$\left[\frac{1}{5}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}+\frac{1}{5}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$A+\frac{1}{5}E=\frac{1}{5}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}+\frac{1}{5}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}\xrightarrow{r_2-r_1,\ r_3-r_1,\frac{5}{2}r_1}$$

$$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$$

同解方程为$x_1+x_2+x_3=0$，得$\left(A+\frac{1}{5}E\right)x=0$ 的基础解系为

$$\xi_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\quad \xi_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

故属于$\lambda_1=\lambda_2=-\frac{1}{5}$ 的正规化后特征向量为

$$\xi_1=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix},\quad \xi_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

当$\lambda_3=1$ 时，将$\lambda_3=1$ 代入$(A-\lambda E)x=0$，即$(A-E)x=0$，亦即

$$\left[\frac{1}{5}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$A-E=\frac{1}{5}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-\frac{4}{5}&\frac{2}{5}&\frac{2}{5}\\\frac{2}{5}&-\frac{4}{5}&\frac{2}{5}\\\frac{2}{5}&\frac{2}{5}&-\frac{4}{5}\end{pmatrix}\xrightarrow{r_2+\frac{1}{2}r_1,\ r_3+\frac{1}{2}r_1}$$

$$\begin{pmatrix}-\frac{4}{5}&\frac{2}{5}&\frac{2}{5}\\0&-\frac{3}{5}&\frac{3}{5}\\0&\frac{3}{5}&-\frac{3}{5}\end{pmatrix}\xrightarrow{r_3+r_2}\begin{pmatrix}-\frac{4}{5}&\frac{2}{5}&\frac{2}{5}\\0&-\frac{3}{5}&\frac{3}{5}\\0&0&0\end{pmatrix}\xrightarrow{-\frac{5}{4}r_1,-\frac{5}{3}r_2}\begin{pmatrix}1&-\frac{1}{2}&-\frac{1}{2}\\0&1&-1\\0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1+\frac{1}{2}r_2}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}$$

同解方程为$\begin{cases}x_1-x_3=0\\x_2-x_3=0\end{cases}$，得$(A-E)x=0$ 的基础解系为$\xi_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$

故属于$\lambda_3=1$ 的正规化后特征向量为$\xi_3=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

(b) $\lim\limits_{n\to\infty}A^n=\lim\limits_{n\to\infty}\left[\frac{1}{5}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}\right]^n=\lim\limits_{n\to\infty}\left(\frac{1}{5}\right)^n\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}^n=\lim\limits_{n\to\infty}\frac{1}{5^n}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}^n=$ 0（分母趋于无穷大时，整个分数等于 0）

2. $M=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$

利用公式法求矩阵 M 的逆阵

$$|M|=\begin{vmatrix}2&1&1\\1&2&1\\1&1&2\end{vmatrix}=2^3+1^3+1^3-1\times2\times1-1\times1\times2-2\times1\times1$$

$$=8+1+1-2-2-2=4$$

$M_{11}=\begin{vmatrix}2&1\\1&2\end{vmatrix}=3$，$M_{12}=-\begin{vmatrix}1&1\\1&2\end{vmatrix}=-1$，$M_{13}=\begin{vmatrix}1&2\\1&1\end{vmatrix}=-1$，

$M_{21}=-\begin{vmatrix}1&1\\1&2\end{vmatrix}=-1$，$M_{22}=\begin{vmatrix}2&1\\1&2\end{vmatrix}=3$，$M_{23}=-\begin{vmatrix}2&1\\1&1\end{vmatrix}=-1$，

$M_{31}=\begin{vmatrix}1&1\\2&1\end{vmatrix}=-1$，$M_{32}=-\begin{vmatrix}2&1\\1&1\end{vmatrix}=-1$，$M_{33}=\begin{vmatrix}2&1\\1&2\end{vmatrix}=3$，

$$M^{-1}=\frac{M^*}{|M|}=\frac{1}{4}\begin{pmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&-\frac{1}{4}&-\frac{1}{4}\\-\frac{1}{4}&\frac{3}{4}&-\frac{1}{4}\\-\frac{1}{4}&-\frac{1}{4}&\frac{3}{4}\end{pmatrix}$$

$$\left|M^{-1}-\lambda E\right|=\begin{vmatrix}\frac{3}{4}-\lambda&-\frac{1}{4}&-\frac{1}{4}\\-\frac{1}{4}&\frac{3}{4}-\lambda&-\frac{1}{4}\\-\frac{1}{4}&-\frac{1}{4}&\frac{3}{4}-\lambda\end{vmatrix}$$

$$=\left(\frac{3}{4}-\lambda\right)^3+\left(-\frac{1}{4}\right)^3+\left(-\frac{1}{4}\right)^3-\left[\left(-\frac{1}{4}\right)\times\left(\frac{3}{4}-\lambda\right)\times\left(-\frac{1}{4}\right)\right]$$
$$-\left[\left(-\frac{1}{4}\right)\times\left(-\frac{1}{4}\right)\times\left(\frac{3}{4}-\lambda\right)\right]-\left[\left(\frac{3}{4}-\lambda\right)\times\left(-\frac{1}{4}\right)\times\left(-\frac{1}{4}\right)\right]$$
$$=-\lambda^3+\frac{9}{4}\lambda^2-\frac{24}{16}\lambda+\frac{16}{64}$$
$$=64\lambda^3-144\lambda^2+96\lambda-16$$
$$=16\cdot(\lambda-1)^2\cdot(4\lambda-1)$$
$$=0$$

解得三个特征值分别为$\lambda_1=\frac{1}{4}$, $\lambda_2=\lambda_3=1$

下面求各个特征值对应的特征向量

当$\lambda_1=\frac{1}{4}$ 时，将$\lambda_1=\frac{1}{4}$代入$(M^{-1}-\lambda E)x=0$，即$\left(M^{-1}-\frac{1}{4}E\right)x=0$，亦即

$$\left[\begin{pmatrix}\frac{3}{4}&-\frac{1}{4}&-\frac{1}{4}\\-\frac{1}{4}&\frac{3}{4}&-\frac{1}{4}\\-\frac{1}{4}&-\frac{1}{4}&\frac{3}{4}\end{pmatrix}-\frac{1}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$M^{-1}-\frac{1}{4}E=\begin{pmatrix}\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4}\end{pmatrix}-\frac{1}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2}\end{pmatrix}\xrightarrow{r_2+\frac{1}{2}r_1,\ r_3+\frac{1}{2}r_1}\begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\\ 0 & \frac{3}{8} & -\frac{3}{8}\\ 0 & -\frac{3}{8} & \frac{3}{8}\end{pmatrix}$$

$$\xrightarrow{r_3+r_2}\begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4}\\ 0 & \frac{3}{8} & -\frac{3}{8}\\ 0 & 0 & 0\end{pmatrix}\xrightarrow{2r_1,\frac{8}{3}r_2}\begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{r_1+\frac{1}{2}r_2}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$$

同解方程为$\begin{cases}x_1-x_3=0\\ x_2-x_3=0\end{cases}$，得$(M^{-1}-E)x=0$ 的基础解系为$\xi_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$

故属于$\lambda_1=\frac{1}{4}$的特征向量为$k_1\xi_1=k_1\begin{pmatrix}1\\1\\1\end{pmatrix}$，其中$k_1$为任意非零实数

因为$v=\begin{pmatrix}1\\ \alpha\\ 1\end{pmatrix}$是$M^{-1}$其中一个特征向量，因此$\lambda_1=\frac{1}{4}$ 时，当$\alpha=1$。

当$\lambda_2=\lambda_3=1$ 时，将$\lambda=1$ 代入$(M^{-1}-\lambda E)x=0$，即$(M^{-1}-E)x=0$，亦即

$$\left[\begin{pmatrix}\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4}\end{pmatrix}-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$M^{-1}-E=\begin{pmatrix}\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4}\end{pmatrix}-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\end{pmatrix}\xrightarrow{r_2-r_1,\ r_3-r_1,-4r_1}\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

同解方程为$x_1+x_2+x_3=0$，得$(M^{-1}-E)x=0$ 的基础解系为

$\xi_2=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$，$\xi_3=\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$

故属于$\lambda_2=\lambda_3=1$ 的特征向量为$k_2\xi_2=k_2\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$，$k_3\xi_3=k_3\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$其中$k_2,k_3$ 为任意非零实数

因为$v=\begin{pmatrix}1\\ \alpha\\ 1\end{pmatrix}$是$M^{-1}$其中一个特征向量，因此$\lambda_2=\lambda_3=1$ 时，v无法满足条件。

综上，$\alpha=1$。

3. 由$x(t)=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$，$\frac{dx(t)}{dt}=-\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}x(t)$可得$\begin{cases}\frac{dx_1}{dt}=-3x_1-x_2\\ \frac{dx_2}{dt}=-x_1-3x_2\end{cases}$

此方程组可化为$\frac{dx}{dt}=Ax$, A $=-\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}=\begin{pmatrix}-3 & -1\\ -1 & -3\end{pmatrix}$

先化 A 为对角矩阵，为此找出 P，使$P^{-1}AP=\Lambda$，求 P 的过程如下：

$$|A-\lambda E|=\begin{vmatrix}-3-\lambda & -1\\ -1 & -3-\lambda\end{vmatrix}=(-3-\lambda)^2-1=0$$

解得两个特征值分别为$\lambda_1=-2,\ \lambda_2=-4$

将$\lambda_1=-2$ 代入$(A+2E)x=0$，

$$A+2E=\begin{pmatrix}-3+2 & -1\\ -1 & -3+2\end{pmatrix}=\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\xrightarrow{r_2-r_1,-r_1}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$$

这时同解方程组为$x_1+x_2=0$，求得$\lambda_1=-2$ 对应的特征向量$\xi_1=\begin{pmatrix}1\\ -1\end{pmatrix}$

将$\lambda_2=-4$ 代入$(A+4E)x=0$，

$$A+4E=\begin{pmatrix}-3+4 & -1\\ -1 & -3+4\end{pmatrix}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\xrightarrow{r_2+r_1}\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$$

这时同解方程组为$x_1 - x_2 = 0$，求得$\lambda_2 = -4$ 对应的特征向量$\xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

故$P = (\xi_1, \xi_2) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$，

$$(P|E) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_2+r_1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}r_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{r_1-r_2}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

因此$P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$，$\Lambda = \begin{pmatrix} -2 & \\ & -4 \end{pmatrix}$

由$x = Py$，$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$，化方程组为

$P\frac{dy}{dt} = APy$, $\frac{dy}{dt} = P^{-1}APy = \Lambda \mathrm{y}$

即$\begin{cases} \frac{dy_1}{dt} = -2y_1 \\ \frac{dy_2}{dt} = -4y_2 \end{cases}$ 解得$\begin{cases} y_1 = c_1 e^{-2t} \\ y_2 = c_2 e^{-4t} \end{cases}$

最后由$x = Py$，即$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$，得到方程组通解

$$\begin{cases} x_1 = y_1 + y_2 = c_1 e^{-2t} + c_2 e^{-4t} \\ x_2 = -y_1 + y_2 = -c_1 e^{-2t} + c_2 e^{-4t} \end{cases}$$

因为 $x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$，代入上式得

$$\begin{cases} 1 = c_1 e^{-2\cdot 0} + c_2 e^{-4\cdot 0} = c_1 + c_2 \\ 2 = -c_1 e^{-2\cdot 0} + c_2 e^{-4\cdot 0} = -c_1 + c_2 \end{cases}$$

解得$c_1 = -\frac{1}{2}$，$c_2 = \frac{3}{2}$

$x(t)$的解析解为

$$\begin{cases} x_1 = -\frac{1}{2} e^{-2t} + \frac{3}{2} e^{-4t} \\ x_2 = \frac{1}{2} e^{-2t} + \frac{3}{2} e^{-4t} \end{cases}$$

4. 空间 V 的基向量$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$，$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$，求$v$向量使得该向量到向量$y = \begin{pmatrix} 8 \\ 0 \\ 6 \end{pmatrix}$的欧式距离最短。

设

$$v=\frac{[v_1,y]}{[v_1,v_1]}v_1+\frac{[v_2,y]}{[v_2,v_2]}v_2$$
$$=\frac{1\times 8+1\times 0+0\times 6}{1\times 1+1\times 1+0\times 0}v_1+\frac{0\times 8+1\times 0+1\times 6}{0\times 0+1\times 1+1\times 1}v_2$$
$$=4\begin{pmatrix}1\\1\\0\end{pmatrix}+3\begin{pmatrix}0\\1\\1\end{pmatrix}$$
$$=\begin{pmatrix}4\\7\\3\end{pmatrix}$$

5. 若u和v, $u+v$和$u-v$均为正交，则

$$[u,v]=0$$
$$[u+v,u-v]=0$$
$$[u]^2+[u,v]-[u,v]-[v]^2=0$$
$$\|u\|^2-\|v\|^2=0$$
$$\|u\|^2=\|v\|^2$$
$$\|v\|=\|u\|=1$$

6. 设对称矩阵$A=\begin{pmatrix}a & b\\ b & c\end{pmatrix}$，则有$\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$，亦即

$$\begin{cases}a-b=3\\ b-c=4\end{cases}$$

解得$b=a-3$，$c=a-7$

$A=\begin{pmatrix}a & a-3\\ a-3 & a-7\end{pmatrix}$，其中 a 为任意数

7. 给出$x_1^2+x_2^2+x_3^2=1$，求$f(x)=x_1^2+2x_2^2+x_3^2-2x_1x_2-2x_2x_3=3$

8. A 为反对称矩阵，则$A^T+A=0$

因为$(I-A^2)^{'}=I^{'}-(A^2)^{'}=I-\left(A^{'}\right)^2=I-(-A)^2=I-A^2$

则$I-A^2$为对称矩阵，而对$\forall x\neq 0$，有

$$x^{'}(I-A^2)x=x^{'}\left(I+A^{'}A\right)x=x^{'}x+x^{'}A^{'}Ax=x^{'}x+(Ax)^{'}(Ax)>0$$

故$I-A^2$是正定矩阵。