6.

$|A| \neq 0$ 时，满足题目要求.

$$|A| = \begin{vmatrix} 1 & 4 & 5 \\ \alpha^2 & 2\alpha & 0 \\ 3 & 2 & 6 \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= 1 \times 2\alpha \times 6 + 0 + 5 \times \alpha^2 \times 2 - 0 - 4 \times \alpha^2 \times 6 - 5 \times 2\alpha \times 3$$

$$= 12\alpha + 10\alpha^2 - 24\alpha^2 - 30\alpha$$

$$= -14\alpha^2 - 18\alpha$$

$$= -\alpha(14\alpha - 18) \neq 0.$$

$\therefore \alpha \neq 0$ 且 $\alpha \neq \frac{9}{7}$ 时，满足条件.

7. $$\begin{pmatrix}4\\10\\1\end{pmatrix}x_1+\begin{pmatrix}-18\\-32\\9\end{pmatrix}x_2=\begin{pmatrix}1\\9\\7\end{pmatrix}$$

$$\overline{A}=(A\,\vdots\,b)=\left(\begin{array}{cc:c}4&-18&1\\10&-32&9\\1&9&7\end{array}\right)\xrightarrow{r_1\leftrightarrow r_3}\left(\begin{array}{cc:c}1&9&7\\10&-32&9\\4&-18&1\end{array}\right)\xrightarrow[r_3-4r_1]{r_2-10r_1}$$

$$\left(\begin{array}{cc:c}1&9&7\\0&-122&-61\\0&-54&-27\end{array}\right)\xrightarrow[-\frac{1}{54}r_3]{-\frac{1}{122}r_2}\left(\begin{array}{cc:c}1&9&7\\0&1&\frac{1}{2}\\0&1&\frac{1}{2}\end{array}\right)\xrightarrow{r_3-r_2}$$

$$\left(\begin{array}{cc:c}1&9&7\\0&1&\frac{1}{2}\\0&0&0\end{array}\right)\xrightarrow{r_1-9r_2}\left(\begin{array}{cc:c}1&0&\frac{5}{2}\\0&1&\frac{1}{2}\\0&0&0\end{array}\right)$$

因为线性方程组经初等变换后成为共同解方程组.

$\therefore\ x_1=\frac{5}{2},\ x_2=\frac{1}{2}$

$\therefore\ \begin{pmatrix}1\\9\\7\end{pmatrix}$ 是这两个向量 $\begin{pmatrix}4\\10\\1\end{pmatrix}$, $\begin{pmatrix}-18\\-32\\9\end{pmatrix}$ 张成的线性空间里.

8. 若向量线性独立，则有 $Ax=0$，当且仅当 $x=0$.

$$|A|=\begin{vmatrix} 1 & \lambda & 1 \\ 0 & 1 & \lambda \\ 0 & -\lambda & 2\lambda+1 \end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-$$

$$a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$$

$$=1\times1\times(2\lambda+1)+\lambda\times\lambda\times0+1\times0\times(-\lambda)-$$

$$1\times\lambda\times(-\lambda)-\lambda\times0\times(2\lambda+1)-1\times1\times0$$

$$=\cancel{2+}\ 2\lambda+1-(-\lambda^2)$$

$$=\lambda^2+2\lambda+1$$

$$=(\lambda+1)^2$$

只有零解的充要条件是 $|A|\neq0$.

$\therefore$ $\lambda\neq-1$ 时，向量线性独立.

9. 假设

$$a=\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix},\ b=\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix},\ c=\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}\ d=\begin{pmatrix}d_1\\d_2\\d_3\\d_4\end{pmatrix}\ \ e=\begin{pmatrix}e_1\\e_2\\e_3\\e_4\end{pmatrix}$$

$$M=(a\,b\,c\,d)=\begin{pmatrix}a_1&b_1&c_1&d_1\\a_2&b_2&c_2&d_2\\a_3&b_3&c_3&d_3\\a_4&b_4&c_4&d_4\end{pmatrix}$$

$$\det(M)=a_1b_2c_3d_4+b_1c_2d_3a_4+c_1d_2a_3b_4+d_1a_2b_3c_4-$$
$$a_1d_2c_3b_4-b_1a_2d_3c_4-c_1b_2a_3d_4-d_1c_2b_3a_4.$$

$$N=(a\,b\,c\,e)=\begin{pmatrix}a_1&b_1&c_1&e_1\\a_2&b_2&c_2&e_2\\a_3&b_3&c_3&e_3\\a_4&b_4&c_4&e_4\end{pmatrix}$$

$$\det(N)=a_1b_2c_3e_4+b_1c_2e_3a_4+c_1e_2a_3b_4+e_1a_2b_3c_4-$$
$$a_1e_2c_3b_4-b_1a_2e_3c_4-c_1b_2a_3e_4-e_1c_2b_3a_4$$

$$M+N=(2a\ \ 2b\ \ 2c\ \ d+e)=\begin{pmatrix}2a_1&2b_1&2c_1&d_1+e_1\\2a_2&2b_2&2c_2&d_2+e_2\\2a_3&2b_3&2c_3&d_3+e_3\\2a_4&2b_4&2c_4&d_4+e_4\end{pmatrix}$$

$$
\begin{aligned}
\det(M+N) = {} & 2a_1 \cdot 2b_2 \cdot 2c_3 \cdot (d_4+e_4) + 2b_1 \cdot 2c_2 \cdot (d_3+e_3) \cdot 2a_4 + \\
& 2c_1 \cdot (d_2+e_2) \cdot 2a_3 \cdot 2b_4 + (d_1+e_1) \cdot 2a_2 \cdot 2b_3 \cdot 2c_4 - \\
& 2a_1 \cdot (d_2+e_2) \cdot 2c_3 \cdot 2b_4 - 2b_1 \cdot 2a_2 \cdot (d_3+e_3) \cdot 2c_4 - \\
& \cancel{d}\, 2c_1 \cdot 2b_2 \cdot 2a_3 \cdot (d_4+e_4) - (d_1+e_1) \cdot 2c_2 \cdot 2b_3 \cdot \cancel{2b_4}\, 2a_4 \\
= {} & 8a_1b_2c_3d_4 + 8a_1b_2c_3e_4 + 8b_1c_2d_3a_4 + 8b_1\cancel{d_3}c_2e_3a_4 \\
& + 8c_1d_2a_3b_4 + 8c_1e_2a_3b_4 + 8d_1a_2b_3c_4 + 8e_1a_2b_3c_4 - \\
& 8a_1d_2c_3b_4 - 8a_1e_2c_3b_4 - 8b_1a_2d_3c_4 - 8b_1a_2e_3c_4 - \\
& 8c_1b_2a_3d_4 - 8c_1b_2a_3e_4 - 8d_1c_2b_3a_4 - 8e_1c_2b_3a_4 \\
= {} & 8\det(M) + 8\det(N)
\end{aligned}
$$

$$\therefore \det(M+N) = 8\det(M) + 8\det(N)$$

10. $A = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & b & 1 \end{pmatrix}$

$\because |A| = 1$

$A_{11} = \begin{vmatrix} 1 & 0 \\ b & 1 \end{vmatrix} = 1$, $A_{12} = \begin{vmatrix} -a & 0 \\ 0 & 1 \end{vmatrix} = -a$, $A_{13} = \begin{vmatrix} -a & 1 \\ 0 & b \end{vmatrix} = -ab$.

$A_{21} = \begin{vmatrix} 0 & 0 \\ b & 1 \end{vmatrix} = 0$, $A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$, $A_{23} = \begin{vmatrix} 1 & 0 \\ 0 & b \end{vmatrix} = b$

$A_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$, $A_{32} = \begin{vmatrix} 1 & 0 \\ -a & 0 \end{vmatrix} = 0$, $A_{33} = \begin{vmatrix} 1 & 0 \\ -a & 1 \end{vmatrix} = 1$

$\therefore A^{-1} = \dfrac{A^*}{|A|} = \begin{pmatrix} 1 & -a & -ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$