

Q2.

1. according to

$$|\psi\rangle = |t\rangle_n = \cos \frac{\theta}{2} |t\rangle + \sin \frac{\theta}{2} e^{i\phi} |-\rangle$$

we have

$$\begin{cases} \cos \frac{\theta}{2} = \frac{1}{2} \\ \sin \frac{\theta}{2} e^{i\phi} = \frac{\sqrt{3}i}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\theta}{2} = \frac{\pi}{3} \\ \phi = \frac{\pi}{4} \end{cases}$$

So

$$\begin{aligned} \cos \theta &= -\frac{1}{2}, & \sin \theta &= \frac{\sqrt{3}}{2} \\ \cos \phi &= 0, & \sin \phi &= 1 \end{aligned}$$

So

$$\begin{aligned} \hat{n} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \\ &= \left(0, \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned} S_{\hat{n}} &= \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{pmatrix} \end{aligned}$$

2. The answer is $|-\rangle_n$,

we have

$$|-\rangle_n = \sin \frac{\theta}{2} |t\rangle - \cos \frac{\theta}{2} e^{i\phi} |-\rangle = \frac{\sqrt{3}}{2} |t\rangle - \frac{i}{2} |-\rangle$$

$$\begin{aligned}
 3. \quad \langle + | - \rangle &= \left(\frac{1}{2}, -\frac{i\sqrt{3}}{2} \right) \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{i}{2} \end{pmatrix} \\
 &= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0
 \end{aligned}$$

Q 3.

1. Assume $|\psi\rangle = |+\rangle_n = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$
 from $\hat{S}_x, \hat{S}_z$ measurements, we know

$$\langle \psi | \hat{S}_x | \psi \rangle = 0$$

$$\langle \psi | \hat{S}_z | \psi \rangle = 0.25 - 0.75 = -0.5$$

where $\hat{S}_x = S_x / \frac{\hbar}{2}$, $\hat{S}_z = S_z / \frac{\hbar}{2}$

$$\text{So, } \begin{cases} \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{i\phi} + \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{-i\phi} = 0 \\ \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = -\frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \cos \theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \\ e^{i\phi} = \pm i \end{cases}$$

$$\text{So } |\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} i \end{pmatrix}$$

2. for $|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2}i \end{pmatrix}$

$$|\langle t_y | \psi \rangle|^2 = \frac{1}{2\sqrt{2}} (1 + \sqrt{3})$$

$$\approx \left(\frac{2.732}{2.828} \right)^2$$

$$\approx 0.933$$

$$|\langle -y | \psi \rangle|^2 = \frac{1}{2\sqrt{2}} (1 - \sqrt{3})$$

$$\approx 0.067$$

So $\uparrow x$ possibility is 0.933

$\downarrow x$ possibility is 0.067.

Consistent!

for $|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2}i \end{pmatrix}$

$$|\langle t_y | \psi \rangle|^2 = 0.067$$

$$|\langle -y | \psi \rangle|^2 = 0.933$$

So it is not consistent.

I guess it is B_x first and the B_z

Q4. In magnetic field

$$\hat{A} = -\vec{\mu} \cdot \vec{B} = -\gamma \vec{S} \cdot \vec{B}$$

$$= -\frac{\hbar\gamma}{2} (B_x \hat{\sigma}_x + B_y \hat{\sigma}_y + B_z \hat{\sigma}_z)$$

here $\gamma = \frac{-e}{2m}$

when apply $\vec{B} = B_z \hat{z}$,

$$\hat{A} = \frac{eh}{4m} B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E_1 = \frac{eh}{4m} B_z \quad E_2 = -\frac{eh}{4m} B_z$$

when apply $\vec{B} = B_x \hat{x}$

$$\hat{A} = \frac{eh}{4m} B_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $E_1 = \frac{eh}{4m} B_x \quad |E_1\rangle = |+\rangle_x$

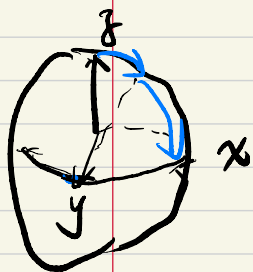
$E_2 = -\frac{eh}{4m} B_x \quad |E_2\rangle = |-\rangle_x$

Start from $|\psi(t=0)\rangle = |+\rangle$
 $= \frac{1}{\sqrt{2}} (|+\rangle_x + |-\rangle_x)$

Then after t , we have

(1) $|\psi(t=t_x)\rangle = \frac{1}{\sqrt{2}} e^{-iE_1 t_x/\hbar} |+\rangle_x + \frac{1}{\sqrt{2}} e^{-iE_2 t_x/\hbar} |-\rangle_x$

To schedule to rotation, we draw the Bloch sphere, blue curve means the rotation.



So we need to rotate to $|+\rangle_y$, then rotate to $|+\rangle_x$.
 with $|-\rangle_y = \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle)$
 $= \frac{1}{2} (|+\rangle_x + |-\rangle_x - i|+\rangle_x + i|-\rangle_x)$
 $= \frac{1-i}{2} |+\rangle_x + \frac{1+i}{2} |-\rangle_x$

So, from (1) we need

$$\frac{1}{\sqrt{2}} e^{-iE_1 t_x/\hbar} = \frac{1-i}{2}$$

So $\frac{E_1 t_x}{\hbar} = \frac{\pi}{4} \Rightarrow t_x = \frac{\pi\hbar}{4E_1}$

Similarly, for the second rotation the rotation angle is still $\frac{\pi}{4}$

So with the same calculation

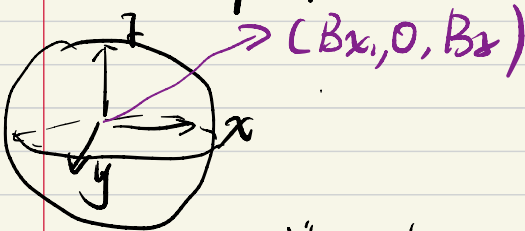
$$t_z = \frac{\pi \hbar}{4 E_1'}$$

Sub in E_1

we have

$$t_x = \frac{\pi m_e}{e B_x} \quad t_z = \frac{\pi m_e}{e B_z}$$

2. Now the rotation is shown in purple line in the Bloch sphere.



The Hamiltonian

$$\hat{H} = \frac{e \hbar}{4 m} \begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}$$

Assume $\frac{B_x}{B_z} = \tan \theta$,

Then we have

$$E_1 = \frac{e\hbar}{4m} B_2 \sqrt{1 + \tan^2 \theta} = \frac{e\hbar}{4m} \frac{B_2}{\cos \theta}$$

$$E_2 = -\frac{e\hbar}{4m} B_2 \sqrt{1 + \tan^2 \theta} = -\frac{e\hbar}{4m} \frac{B_2}{\cos \theta} = -E_1$$

$$|E_1\rangle = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} |-\rangle$$

$$|E_2\rangle = \sin \frac{\theta}{2} |+\rangle - \cos \frac{\theta}{2} |-\rangle$$

With $|\psi(t=0)\rangle = |+\rangle = \cos \frac{\theta}{2} |E_1\rangle + \sin \frac{\theta}{2} |E_2\rangle$

So $|\psi(t)\rangle = \cos \frac{\theta}{2} e^{-iE_1 t/\hbar} |E_1\rangle + \sin \frac{\theta}{2} e^{\frac{iE_1 t}{\hbar}} |E_2\rangle$

We need $|\psi(t)\rangle = |+\rangle$

So $\cos \frac{\theta}{2} e^{-iE_1 t/\hbar} + \sin \frac{\theta}{2} e^{\frac{iE_1 t}{\hbar}} = \frac{1}{2} \psi$

$\cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{-\frac{iE_1 t}{\hbar}} - e^{\frac{iE_1 t}{\hbar}}) = \frac{1}{2} \psi$

Since the left part of ψ is purely imaginary, we need to rewrite the

equation to

$\cos \frac{\theta}{2} e^{-iE_1 t/\hbar} + \sin \frac{\theta}{2} e^{\frac{iE_1 t}{\hbar}} = \frac{i}{2} \psi$

$\cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{-\frac{iE_1 t}{\hbar}} - e^{\frac{iE_1 t}{\hbar}}) = \frac{i}{2} \psi$

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for CD^{θ} , we need the left part to be imaginary, so we have.

$$e^{iE_1 t/\hbar} = i$$

$$\Rightarrow E_1 t/\hbar = \frac{\pi}{2}$$

$$t = \frac{\pi\hbar}{2} \cdot \frac{4m\cos\theta}{e\hbar B_z} = \frac{2\pi m\cos\theta}{eB_z}$$

meanwhile, we need

$$2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta = \frac{1}{\sqrt{2}},$$

So $\theta = \frac{\pi}{4}$, which means we require

$$B_z = B_x$$

$$\text{So } t = \frac{\sqrt{2}\pi m}{eB_z}$$

3. according to 1, 2,

If we do 1. experiment

$$t' = t_0 + t_1 = \frac{2\pi m}{eB_z}$$

If we do 2. experiment

$$t^2 = t = \frac{\sqrt{2}\pi m}{eB_z} < t'$$

So the second is quicker.