# Q1.

1. $\sqrt{5^2+|e^{i\pi/3}|^2} = \sqrt{26}$

So normalized $|\psi\rangle^* = \frac{5}{\sqrt{26}}|+\rangle_x + \frac{e^{i\pi/3}}{\sqrt{26}}|-\rangle_x$

2. rewrite in $S_z$ basis

$$|\psi\rangle^* = \frac{5+e^{\frac{i\pi}{3}}}{2\sqrt{13}}|+\rangle + \frac{5-e^{\frac{i\pi}{3}}}{2\sqrt{13}}|-\rangle$$

So detect $+\frac{\hbar}{2}$ possibility is

$$\left|\frac{5+e^{\frac{i\pi}{3}}}{2\sqrt{13}}\right|^2 = \frac{31}{52}$$

detect $-\frac{\hbar}{2}$ - - - - - -

$$\left|\frac{5-e^{\frac{i\pi}{3}}}{2\sqrt{13}}\right|^2 = \frac{21}{52}$$

3. $\langle S_z\rangle = \langle\psi^*|S_z|\psi^*\rangle = \left(\frac{31}{52} - \frac{21}{52}\right)\frac{\hbar}{2} = \frac{11}{52}\cdot\frac{\hbar}{2}$

the expectation value is the $+\frac{\hbar}{2}$ possibility minus $-\frac{\hbar}{2}$ possibility

4. $|\psi\rangle^*$ is already in $S_x$ basis

So detect $+\frac{\hbar}{2}$ possibility is $\left|\frac{5}{\sqrt{26}}\right|^2 = \frac{25}{26}$

- - - - - - $-\frac{\hbar}{2}$ - - - - is $\left|\frac{e^{\frac{i\pi}{3}}}{\sqrt{26}}\right|^2 = \frac{1}{26}$

5. $\langle S_x \rangle = \langle \psi^* | S_x | \psi^* \rangle$

$$= \left(\frac{25}{26} - \frac{1}{26}\right)\frac{\hbar}{2}$$

$$= \frac{6}{13}\hbar$$

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S \rangle^2}$$

$$\langle S_x^2 \rangle = \langle \psi^* | S_x^2 | \psi^* \rangle$$

Since $S_x^2 = I\hbar^2$

so $\langle S_x^2 \rangle = \hbar^2$

So $\Delta S_x = \hbar \sqrt{1 - \left(\frac{6}{13}\right)^2}$

$$= \frac{\hbar}{13} \cdot \sqrt{133}$$

Q2.

1. according to

$$|\psi\rangle = |+\rangle_{\hat{n}} = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

we have

$$\begin{cases} \cos\frac{\theta}{2} = \frac{1}{2} \\ \sin\frac{\theta}{2}e^{i\phi} = \frac{\sqrt{3}i}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\theta}{2} = \frac{\pi}{3} \\ \phi = \frac{\pi}{4} \end{cases}$$

So $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\phi = 0$ $\quad \sin\phi = 1$

So $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$= \left(0, \frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

$$S_{\vec{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{pmatrix}$$

2. The answer is $|-\rangle_n$,

we have

$$|-\rangle_n = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle = \frac{\sqrt{3}}{2}|+\rangle - \frac{i}{2}|-\rangle$$

3. $\langle +|-\rangle$

$$= \left(\frac{1}{2}, -\frac{i\sqrt{3}}{2}\right)\begin{pmatrix}\frac{\sqrt{3}}{2} \\ -\frac{i}{2}\end{pmatrix}$$

$$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

Q3.

1. Assume $|\psi\rangle = |+\rangle_n = \begin{pmatrix}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$

from $\hat{x}$, $\hat{z}$ measurements, we know

$$\langle\psi|\hat{\sigma}_x|\psi\rangle = 0$$

$$\langle\psi|\hat{\sigma}_z|\psi\rangle = 0.25 - 0.75 = -0.5$$

where $\hat{\sigma}_x = S_x/\frac{\hbar}{2}$, $\sigma_z = S_z/\frac{\hbar}{2}$

So.
$$\begin{cases}\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} = 0 \\ \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = -\frac{1}{2}\end{cases}$$

$$\Rightarrow \begin{cases}\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3} \\ e^{i\phi} = \pm i\end{cases}$$

So $|\psi\rangle = \begin{pmatrix}\frac{1}{2} \\ \frac{\sqrt{3}}{2}i\end{pmatrix}$ or $\begin{pmatrix}\frac{1}{2} \\ -\frac{\sqrt{3}}{2}i\end{pmatrix}$

2. for $|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2}i \end{pmatrix}$

$$|\langle +y|\psi\rangle|^2 = \frac{1}{2\sqrt{2}}(1+\sqrt{3})$$

$$\approx \left(\frac{2.732}{2.828}\right)^2$$

$$\approx 0.933$$

$$|\langle -y|\psi\rangle|^2 = \frac{1}{2\sqrt{2}}(1-\sqrt{3})$$

$$\approx 0.067$$

So $\uparrow_y$ possibility is 0.933

$\downarrow_y$ possibility is 0.067.

Consistent!

for $|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2}i \end{pmatrix}$

$$|\langle +y|\psi\rangle|^2 = 0.067$$

$$|\langle -y|\psi\rangle|^2 = 0.933$$

So it is not consistent.

I guess it is $B_x$ first and the $B_z$

Q4. In magnetic field

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B}$$

$$= -\frac{\hbar\gamma}{2}(B_x\hat{\sigma}_x + B_y\hat{\sigma}_y + B_z\hat{\sigma}_z)$$

here $\gamma = \frac{-e}{2m}$

when apply $\vec{B} = B_z\hat{z}$,

$$\hat{H} = \frac{e\hbar}{4m}B_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$E_1' = \frac{e\hbar}{4m}B_z$ $\quad E_2' = -\frac{e\hbar}{4m}B_z$

when apply $\vec{B} = B_x\hat{x}$

$$\hat{H} = \frac{e\hbar}{4m}B_x\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

So $E_1 = \frac{e\hbar}{4m}B_x$ $\quad |E_1\rangle = |+\rangle_x$

$E_2 = -\frac{e\hbar}{4m}B_x$ $\quad |E_2\rangle = |-\rangle_x$

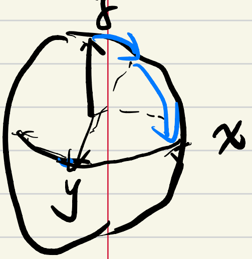
Start from $|\psi(t=0)\rangle = |+\rangle$

$$= \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)$$

Then after $t_1$, we have

(I) $|\psi(t=t_x)\rangle = \frac{1}{\sqrt{2}} e^{-iE_1 t_x/\hbar}|+\rangle_x + \frac{1}{\sqrt{2}} e^{-iE_2 t_x/\hbar}|-\rangle_x$

To schedule to rotation, we draw the block sphere, blue curve means the rotation.




So we need to rotate to $|+\rangle_y$, then rotate to $|+\rangle_x$.

with $|-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$

$$= \frac{1}{2}(|+\rangle_x + |-\rangle_x - i|+\rangle_x + i|-\rangle_x)$$

$$= \frac{1-i}{2}|+\rangle_x + \frac{1+i}{2}|-\rangle_x$$

So from (I) we need

$$\frac{1}{\sqrt{2}} e^{-iE_1 t_x/\hbar} = \frac{1-i}{2}$$

So $\frac{E_1 t_x}{\hbar} = \frac{\pi}{4} \Rightarrow t_x = \frac{\pi\hbar}{4E_1}$

Similarly, for the second rotation the rotation angle is still $\frac{\pi}{4}$

So with the same calculation

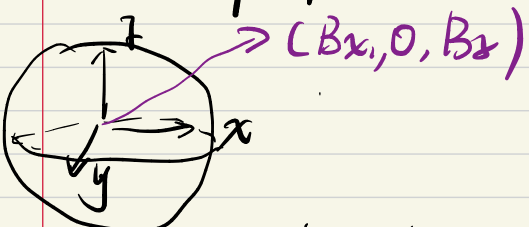
$$t_z = \frac{\pi\hbar}{4E_1'}$$

sub in $E_1$

we have

$$t_x = \frac{\pi m_e}{eB_x} \qquad t_z = \frac{\pi m_e}{eB_z}$$

2. Now the rotation is shown in purple line in the Bloch sphere.




The Hamiltonian

$$\hat{H} = \frac{e\hbar}{4m}\begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}$$

Assume $\frac{B_x}{B_z} = \tan\theta$,

Then we have

$$E_1 = \frac{e\hbar}{4m} B_z \sqrt{1+\tan^2\theta} = \frac{e\hbar}{4m}\frac{B_z}{\cos\theta}$$

$$E_2 = -\frac{e\hbar}{4m} B_z \sqrt{1+\tan^2\theta} = -\frac{e\hbar}{4m}\frac{B_z}{\cos\theta} = -E_1$$

$$|E_1\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$$

$$|E_2\rangle = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle$$

with $|\psi(t=0)\rangle = |+\rangle = \cos\frac{\theta}{2}|E_1\rangle + \sin\frac{\theta}{2}|E_2\rangle$

So $|\psi(t)\rangle = \cos\frac{\theta}{2} e^{-iE_1t/\hbar}|E_1\rangle + \sin\frac{\theta}{2} e^{\frac{iE_1t}{\hbar}}|E_2\rangle$

We need $|\psi(t)\rangle = |+\rangle_x$

So

$$\begin{cases} \cos^2\frac{\theta}{2}\, e^{-iE_1t/\hbar} + \sin^2\frac{\theta}{2}\, e^{\frac{iE_1t}{\hbar}} = \frac{1}{\sqrt{2}} & ① \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{\frac{-iE_1t}{\hbar}} - e^{\frac{iE_1t}{\hbar}}\right) = \frac{1}{\sqrt{2}} & ② \end{cases}$$

since the left part of ② is purely imaginary, we need to rewrite the equation to

$$\begin{cases} \cos^2\frac{\theta}{2}\, e^{-iE_1t/\hbar} + \sin^2\frac{\theta}{2}\, e^{\frac{iE_1t}{\hbar}} = \frac{i}{\sqrt{2}} & ①^* \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{\frac{-iE_1t}{\hbar}} - e^{\frac{iE_1t}{\hbar}}\right) = \frac{i}{\sqrt{2}} & ②^* \end{cases}$$

for (1)$^{\circledast}$, we need the left part to be imaginary, so we have.

$$e^{iE_1 t/\hbar} = i$$

$$\Rightarrow \quad E_1 t/\hbar = \frac{\pi}{2}$$

$$t = \frac{\pi\hbar}{2} \cdot \frac{4m\cos\theta}{e\hbar B_z} = \frac{2\pi m\cos\theta}{eB_z}$$

meanwhile, we need

$$2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta = \frac{1}{\sqrt{2}},$$

So $\theta = \frac{\pi}{4}$, which means we require $B_z = B_x$

So $t = \dfrac{\sqrt{2}\pi m}{eB_z}$

3. according to 1, 2,

If we do 1. experiment

$$t' = t_0 + t_1 = \frac{2\pi m}{eB_z}$$

If we do 2. experiment

$$t^2 = t = \frac{\sqrt{2}\pi m}{eB_z} < t'$$

So the second is quicker.