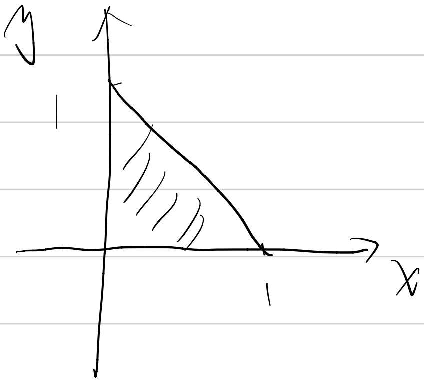
Q4

(a)

$0 < x, y < 1$, $x + y < 1$



(b)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$$

$$\int_0^1 \int_0^{1-x} c\, x^2 y \, dy\, dx = 1$$

$$c \int_0^1 \frac{1}{2} x^2 (1-x)^2 \, dx = 1$$

$$\frac{1}{2} c \int_0^1 x^4 - 2x^3 + x^2 \, dx = 1$$

$$\frac{1}{2} c \left(\frac{1}{5} - 2\frac{1}{4} + \frac{1}{3} \right) = 1$$

$$c = 60$$

(c)

The marginal pdf for X.

$$f_X(x) = \int_0^{1-x} f(x,y)\,dy$$

$$= 60x^2 \int_0^{1-x} y\,dy$$

$$= 30x^2(1-x)^2 \qquad 0 \le x \le 1$$

The marginal pdf for Y

$$f_Y(y) = \int_0^{1-y} f(x,y)\,dx$$

$$= 60y \int_0^{1-y} x^2\,dx$$

$$= 20y(1-y)^3 \qquad 0 \le y \le 1.$$

(d)

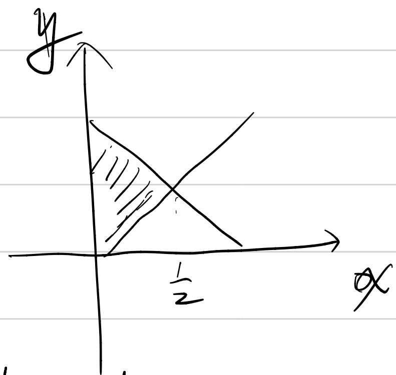
$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{60x^2y}{20y(1-y)^3} \qquad 0 < x < 1-y$$

$$= \frac{3x^2}{(1-y)^3} \qquad 0 < x < 1-y.$$

(e) X and Y are not independent since

$$f(x,y) \neq f_X(x)\, f_Y(y).$$

(f)



$P(X \le Y)$

$$= \int_0^{\frac{1}{2}} \int_x^{1-x} 60x^2 y \, dy \, dx$$

$$= 30 \int_0^{\frac{1}{2}} x^2 \left[(1-x)^2 - x^2 \right] dx$$

$$= 30 \int_0^{\frac{1}{2}} x^2 (1-2x)\, dx$$

$$= 30 \left(\frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 - \frac{2}{4} \cdot \left(\frac{1}{2}\right)^4 \right) = 0.3125$$

(g)

$P(X \le 0.4 \mid Y \ge 0.5)$

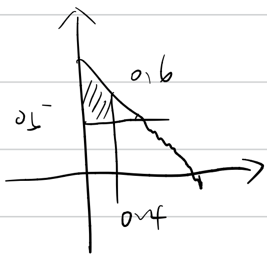
$$= \frac{P(X \le 0.4,\ Y \ge 0.5)}{P(Y \ge 0.5)}$$

$$P(Y \ge 0.5) = \int_{\frac{1}{2}}^{1} 20y(1-y)^3 \, dy$$

$$= 20 \int_{\frac{1}{2}}^{1} y(1-3y+3y^2-y^3) \, dy$$

$$= 20\left(\frac{1}{2} \cdot \frac{3}{4} - 3 \cdot \frac{1}{3} \cdot \frac{7}{8} + 3 \cdot \frac{1}{4} \cdot \frac{15}{16} - \frac{1}{5} \cdot \frac{31}{32} \right)$$

$$= 0.1875$$



$P(X \le 0.4,\ Y \ge 0.5)$

$$= \int_{0}^{0.4} \int_{0.5}^{1-x} 60 x^2 y \, dy \, dx$$

$$= 60 \int_0^{0.4} \frac{1}{2} x^2 \left((1-x)^2 - \frac{1}{4} \right) dx$$

$$= 0.15744$$

$$P(X \le 0.4 \mid Y \ge 0.25) = \frac{0.15744}{0.1875}$$

$$= 0.83968$$