We claim that the identity map ι in S that $\iota(s) \coloneqq s$ for $s \in S$ is the bijection. This is done by showing that the coproduct of all preimages is isomorphic to the domain.

First, we show that $f^{-1}(t_1)$and $f^{-1}(t_2)$ are disjointed if $t_1 \neq t_2$. If one of them is empty (in the case it is not in the image of f), they are obviously disjointed, since empty set does not have non-empty intersection with any other set. If they are both non-empty, assume they have a non-empty intersection $f^{-1}(t_1) \cap f^{-1}(t_2)$. Then we can take at least an element s from this intersection, that $f(s)$ is mapped to t_1 and t_2 at the same time, violating the definition of map (one element in the domain is mapped to one and only one element in the range).

By far we have shown that all the preimages are disjointed. It naturally follows that the coproduct of them are isomorphic to the domain S itself, thus we have the identity map as a bijection (possibly composing the isomorphism between S and $\coprod_{t \in T} f^{-1}(t)$, but this isomorphism is essentially also an identity mapping after canonical projection).