(a) If B is a basis for $M_{2\times 2}(R)$, it means that

1. B_1, B_2, B_3, B_4 are linearly independent.

2. $\text{span}(B) = M_{2\times 2}(R)$.

Step 1. Let $c_1B_1 + c_2B_2 + c_3B_3 + c_4B_4 = 0$

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1+c_2+c_3+c_4 & c_1+c_2+c_3 \\ c_1+c_2 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So we have $c_1 = c_2 = c_3 = c_4 = 0$

B_1, B_2, B_3, B_4 are linearly independent.

Step 2.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

For any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M_{2\times 2}(R)$,

we can find constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \alpha_4 B_4$$

$$\Rightarrow \begin{aligned} \alpha_1+\alpha_2+\alpha_3+\alpha_4 &= a \\ \alpha_1+\alpha_2+\alpha_3 &= b \\ \alpha_1+\alpha_2 &= c \\ \alpha_1 &= d \end{aligned}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} d & c-d \\ b-c & a-b \end{bmatrix}$$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = dB_1 + (c-d)B_2 + (b-c)B_3 + (a-b)B_4$

that is $\text{span}(B) = M_{2\times 2}(\mathbb{R})$. q.e.d.

(b) To find $[A]_B$.

let $a_1B_1 + a_2B_2 + a_3B_3 + a_4B_4 = A$

$$\begin{bmatrix} a_1+a_2+a_3+a_4 & a_1+a_2+a_3 \\ a_1+a_2 & a_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

we have $a_1 = 4$,

$a_2 = a_3 = a_4 = -1$

$$[A]_B = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & -1 \end{bmatrix}$$