a) i. $x_p(t) = x_0$

$\dot{x}_p(t) = 0$

$\ddot{x}_p(t) = 0 \quad \Rightarrow \quad 0 + w^2 x_p(t) = a(t)$

$w^2 x_0 = a(t)$

$x_0 = \frac{a(t)}{w^2} = \frac{a_0}{w^2}$

ii. $x(t) = C_1 \cos wt + C_2 \sin wt$ + particular solution

$= C_1 \cos wt + C_2 \sin wt + \frac{a}{w^2}$

$x(0) = 0 \Rightarrow C_1 + \frac{a}{w^2} = 0$
$x'(0) = 0 \Rightarrow C_2 w \cos wt = 0$
$\Rightarrow \begin{cases} C_1 = -\frac{a}{w^2} \\ C_2 = 0 \end{cases}$

Thus $x(t) = \frac{a - a\cos wt}{w^2}$

b. i. $x(t) = x_1(t) + x_2(t)$

$x(0) = x_1(0) + x_2(0) = 0$

$\dot{x}(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0$

$\left.\begin{aligned} \ddot{x}_1 + w^2 x_1 &= f_1 \\ \ddot{x}_2 + w^2 x_2 &= f_2 \end{aligned}\right\}$ summation

$\ddot{x}_1 + \ddot{x}_2 + w^2(x_1 + x_2) = f_1 + f_2$

$$(x_1+x_2)'' + w^2(x_1+x_2) = f_1+f_2$$

$$x'' + w^2 x = f_1+f_2$$

ii) for $0 < t < t_0$

$$x(t) = \frac{a_0 - a_0\cos wt}{w^2} \qquad x'(t) = \frac{a_0 w \sin wt}{w^2}$$

$$= \frac{a_0}{w}\sin wt$$

For $t > t_0$

$x(t)$ is a solution with

$$x(t_0) = \frac{a_0 - a_0\cos wt_0}{w^2}$$

$$x'(t_0) = \frac{a_0}{w}\sin wt_0$$

$$x(t) = \frac{a_0(\cos(t-t_0)w - \cos wt)}{w^2}$$

In summary

$$x(t) = \begin{cases} 0 & t < 0 \\ \dfrac{a_0(1-\cos wt)}{w^2} & 0 \leq t \leq t_0 \\ \dfrac{a_0(\cos w(t-t_0) - \cos wt)}{w^2} & t > t_0 \end{cases}$$

c. $\boxed{\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}$

$\cos w(t-t_0) - \cos wt$

$= -2 \sin\left(\frac{2wt - t_0}{2}\right) \sin\left(-\frac{wt_0}{2}\right)$

t_0 is very small, we ignore it

sin(very small) $\approx$ very small

$\approx -2\sin(wt)\left(-\frac{wt_0}{2}\right)$

Thus $x(t) = wt_0 \sin(wt) \frac{a_0}{w^2}$

$= \frac{a_0 t_0}{w} \sin(wt)$

d) i $x(t) = \frac{a(t_n)}{w}(t-t_n)\sin(w(t-t_n))$

ii $x(t) = \sum_{n=-\infty}^{\infty} \frac{a(t_n)}{w}(t-t_n)\sin(w(t-t_n))$

iii $G(t,t') = \begin{cases} \frac{1}{mw}\sin[w(t-t')] & t > t' \\ 0 & t < t' \end{cases}$

$m\ddot{x} + mw^2 x = F(t)$

$\ddot{x} + w^2 x = \frac{F(t)}{m}$ (把$\frac{1}{m}$给 G)

$$F(t) = \sum_{n=-\infty}^{\infty} a(t_n)\,(t-t_n)$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{a(t_n)}{w}(t-t_n)\sin(w(t-t_n))}$$

$$x(t) = \int_{-\infty}^{t} F(t)\; G(t, t')\,dt'$$

iv.
$$x(t) = \int_0^t F_0 e^{-rt}\frac{1}{mw}\sin[w(t-t')]\,dt'$$

$$= F_0 e^{-rt}\frac{1}{mw}\int_0^t \sin[w(t-t')]\,dt'$$

$$= F_0 e^{-rt}\frac{1}{mw}\frac{1-\cos wt}{w}$$

$$= -\frac{e^{-rt}F_0(-1+\cos wt)}{mw^2}$$