$$f(x+y) = e^x f(y) + e^y f(x)$$

(a)

$$f(x+h) = e^x f(h) + e^h f(x)$$

$$\frac{f(x+h) - f(x)}{h} = \frac{e^x f(h)}{h} + \frac{e^h - 1}{h} \cdot f(x)$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} e^x \cdot \frac{f(h)}{h}$$

$$+ \lim_{h \to 0} f(x) \cdot \frac{e^h - 1}{h}$$

$$f'(x) = e^x \cdot f'(0) + f(x) \cdot 1$$

$$= 2022\, e^x + f(x)$$

b) $f'(x) = 2022e^x + f(x)$

$f'(x) - f(x) = 2022e^x$.

The solution is

$f(x) = c \cdot e^x + 2022 \cdot e^x \cdot x$

Recall that $f(x+y) = e^x f(y) + e^y f(x)$

Take $x = y = 0$ $\quad f(0) = 2f(0)$

$f(0) = 0$.

Thus

$c \cdot e^0 + 0 = 0$. $\quad c = 0$.

Thus

$f(x) = 2022 \cdot e^x \cdot x$