

$$a) \quad dX_t = \mu X_t dt + \sigma X_t dW_t$$

$$Y_t = X_t^3$$

$$\frac{\partial Y_t}{\partial X_t} = 3 X_t^2$$

$$\frac{\partial^2 Y_t}{\partial X_t^2} = 6 X_t$$

$$\frac{\partial Y_t}{\partial t} = 0$$

By Itô's lemma

$$dY_t = \left(\frac{\partial Y_t}{\partial t} + \mu X_t \cdot \frac{\partial Y_t}{\partial X_t} + \frac{(\sigma X_t)^2}{2} \cdot \frac{\partial^2 Y_t}{\partial X_t^2} \right) dt$$

$$+ (\sigma X_t) \cdot \frac{\partial Y_t}{\partial X_t} dW_t$$

$$= \left(\mu X_t \cdot (3 X_t^2) + \frac{\sigma^2}{2} X_t^2 \cdot (6 X_t) \right) dt$$

$$+ (6 X_t) \cdot (3 X_t^2) dW_t$$

$$= X_t^3 (3\mu + 3\sigma^2) dt + X_t^3 (36) dW_t$$

Thus

$$\frac{dY_t}{Y_t} = 3(\mu + \sigma^2) dt + 3\sigma dW_t,$$

Y_t is still GBM with a different
mean rate and variance rate

b)

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right)$$

$$\mu = 3.7 \% \quad \sigma = 0.3$$

$$P(S_t \leq x)$$

$$= P \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \leq \log \left(\frac{x}{S_0} \right) \right)$$

1 month

$$P(S_1 \leq x) = P \left(W_t \leq \frac{\log \left(\frac{x}{S_0} \right) + 0.008}{0.3} \right)$$

$$= \Phi \left(\frac{\log \frac{x}{S_0} + 0.008}{0.3} \right)$$

$$P(S_7 \leq x) = P \left(W_t \leq \frac{\log \left(\frac{x}{S_0} \right) + 0.056}{0.3} \right)$$

$$= \Phi \left(\frac{\log \frac{x}{S_0} + 0.056}{0.3 \sqrt{7}} \right)$$

$$E S_7 = S_0 e^{\mu t} = 0.4 \exp(0.037 \times 7) \\ = 0.5183$$

$$\log \frac{S_7}{S_0} = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \\ = 0.056 + 0.3 W_t$$

88% CI for $\log \frac{S_7}{S_0}$ is

$$[0.056 - 0.3 \times 4.11, \quad 0.056 + 0.3 \times 4.11]$$

$$= [-1.1781, \quad 1.2901]$$

Thus 88% CI for S_7 is

$$[0.1231, \quad 1.4532]$$

$$X_1 = Z_1$$

$$X_2 = aZ_1 + \sqrt{1-a^2} Z_2$$

Excel code:

First generate Z_1, Z_2 `NORMINV(RAND(), 0, 1)`

Then use linear transformation to obtain X_1, X_2

When we use 600 observations, the mean and variance are closer to the theoretical mean and variance

d) Theoretical correlation.

$$\text{Cov}(X_1, X_2)$$

$$= \text{cov}(Z_1, aZ_1 + \sqrt{1-a^2}Z_2)$$

$$= a \text{var}(Z_1)$$

$$= a = 0.27$$

only the sample correlation for "sample 2"
is close to the theoretical one.

When sample size is large, the sample
correlation will be close to theoretical
correlation

e)

The fitted line is $X_1 = \phi X_2$, where

$$\phi = \underset{\phi}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^n (X_{1i} - \phi X_{2i})^2.$$

$$\begin{aligned} \phi &= \frac{\operatorname{cov}(X_1, X_2)}{\operatorname{var} X_1} = \rho \cdot \frac{\sqrt{\operatorname{var} X_1 \operatorname{var} X_2}}{\operatorname{var} X_1} \\ &= \rho \cdot \sqrt{\frac{\operatorname{var} X_2}{\operatorname{var} X_1}} \end{aligned}$$

For sample 1, this is

$$\begin{aligned} \phi &= 0.4356 \sqrt{\frac{0.8573}{0.9798}} \\ &= 0.4306 \end{aligned}$$

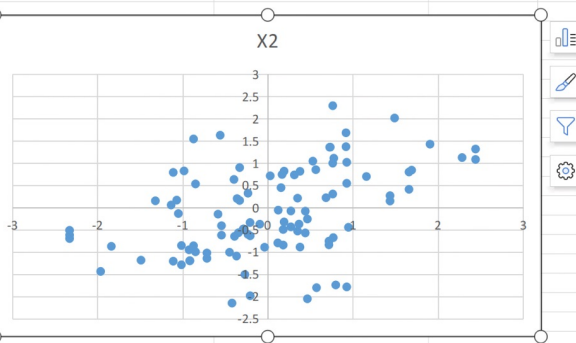
For sample 2,

$$\begin{aligned} \phi &= 0.26345 \sqrt{\frac{1.0389}{1.0098}} \\ &= 0.2674. \end{aligned}$$

Theoretically, $\text{var}(X_1) = \text{var}(X_2) = 1$.

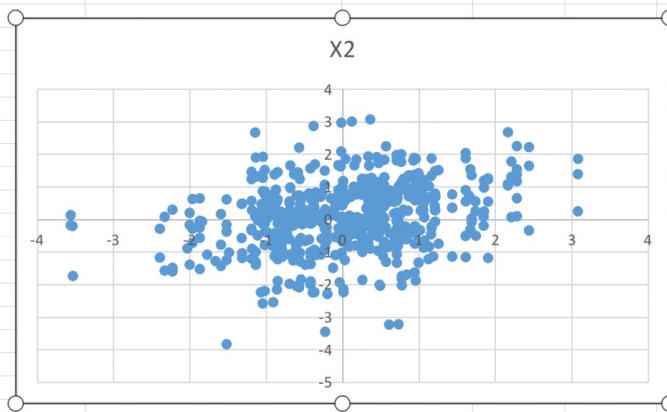
Thus $\phi = \rho$.

Z1	Z2	X1	X2	Mean and Variance of X1	Mean of Variance of X2	rho
-0.448740494	-0.916856704	-0.448740494	-1.003964875	0.03226994	-0.095745274	0.435686435
-0.9831593	1.133487832	-0.9831593	0.825937448	0.979757749	0.957282621	
0.763596226	2.165944464	0.763596226	2.291672966			
0.79879649	-2.029984058	0.79879649	-1.738916052			
0.927817763	0.307419654	0.927817763	0.546512983			
-0.714630032	-0.85778501	-0.714630032	-1.01887726			
-0.206742471	-2.000320418	-0.206742471	-1.981849629			
-0.360352155	0.312484876	-0.360352155	0.203584207			
1.906203254	0.947729403	1.906203254	1.427205917			
-1.319757913	0.529665704	-1.319757913	0.153659454			
1.158999898	0.401859363	1.158999898	0.699864409			
-0.287677313	-0.419580103	-0.287677313	-0.481669908			
-1.135118675	0.378932277	-1.135118675	0.058376812			
-0.924479251	-0.725391881	-0.924479251	-0.948060458			
0.168266826	0.72761414	0.168266826	0.746022828			
-1.110238034	1.133537539	-1.110238034	0.791674051			
-2.327499907	0.011603547	-2.327499907	-0.61725238			
0.350424532	0.123154321	0.350424532	0.213195033			
0.572956206	-2.030349622	0.572956206	-1.800244915			
-1.83727724	-0.388497546	-1.83727724	-0.870133727			
0.181158649	-0.561465269	0.181158649	-0.491699795			
-0.093424314	-0.360364297	-0.093424314	-0.372205048			
-0.558669376	1.847989918	-0.558669376	1.628515436			
0.312489428	0.676282899	0.312489428	0.735538115			
0.71861645	-1.069399476	0.71861645	-0.835655882			



Sample
1

Z2	X1	X2	Mean and Variance of X1	Mean of Variance of X2	rho
0.775850876	-1.060146001	0.460796604	0.057288961	0.06497087	0.26345114
-0.52540216	0.181183694	-0.456969295	1.009738629	1.038896603	
-0.200336486	-1.963097998	-0.722932513			
1.613236922	1.166137737	1.868179011			
-0.486992532	0.947712572	-0.213023392			
-0.123541748	-1.013995439	-0.392732216			
0.77634284	0.246669311	0.8141110431			
0.548285029	-0.871715631	0.29255868			
0.381401284	-2.000717924	-0.172957677			
-1.117168727	-0.093424486	-1.100902051			
-0.28153604	-0.916830882	-0.51862422			
0.798848502	-0.857727878	0.537592999			
-0.725358592	1.847844653	-0.199500951			
-2.000811574	0.84181458	-1.69921214			
0.735457872	0.630407147	0.878353133			
-1.771613109	1.90641336	-1.191084362			
0.337529236	-0.691846071	0.13819507			
0.246723467	-0.491163657	0.10494605			
0.548271777	0.011569101	0.531032798			
-0.691823066	0.825174523	-0.443331859			
0.337510841	-0.857732884	0.093387919			
1.0160267	2.44185121	1.637591622			
-0.635611733	1.01608931	-0.337661205			
0.943818508	0.398239968	1.016290184			
0.943881028	-0.437476595	0.79070691			



Sample
2