

Sampling

2022-05-11

1. Rejection sampling

a) The range of uniform distribution is finite like $[0, M]$ and the range of Weibull distribution is $x \geq 0$. If we use the uniform distribution as the proposal, we can only explore the range of $(0, M)$.

b) Note that, the integration of a density function is 1, thus,

$$A \int_0^3 \frac{k}{\lambda} (x/\lambda)^{k-1} \exp(-(x/\lambda)^k) dx = 1.$$

Also,

$$\frac{k}{\lambda} (x/\lambda)^{k-1} \exp(-(x/\lambda)^k), x \geq 0$$

is the density function of the weibull distribution, thus,

$$\frac{k}{\lambda} (x/\lambda)^{k-1} \exp(-(x/\lambda)^k) dx = F_X(3).$$

Therefore,

$$A = \frac{1}{F_X(3)}.$$

c) Note that, the cumulative distribution of Weibull distribution is

$$F_X(x) = 1 - \exp(-(x/\lambda)^k)$$

Thus,

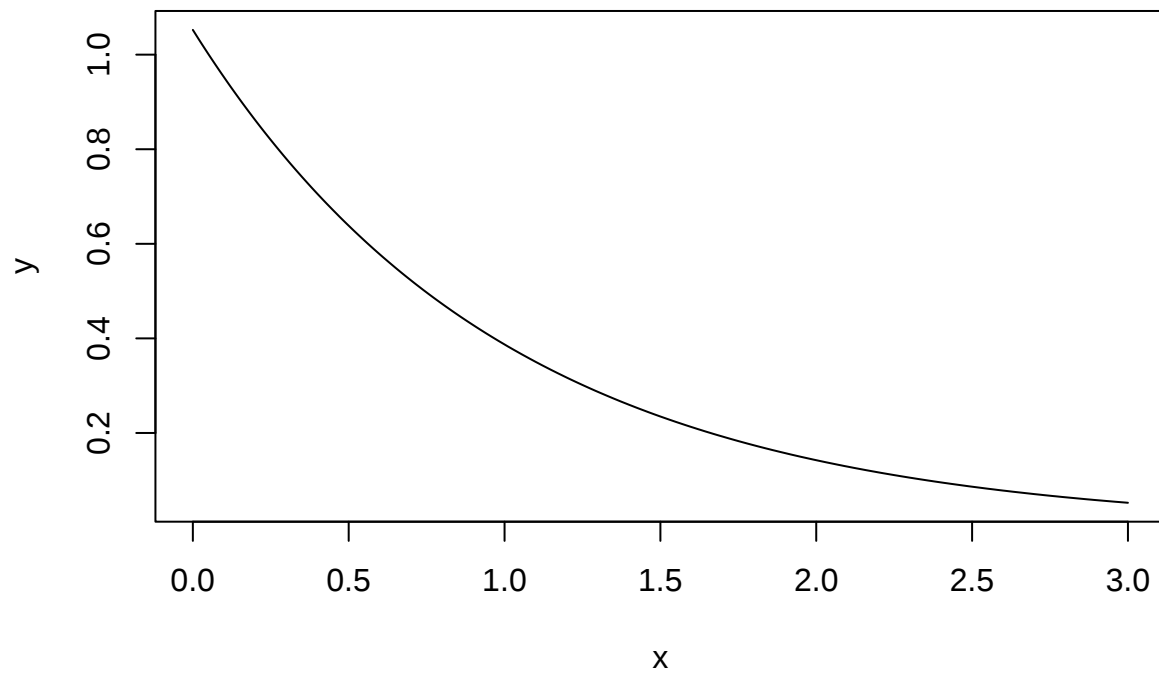
$$F_X(3) = 1 - \exp(-(3/\lambda)^k)$$

```
truncatedweibullpdf <- function(x,k,lambd){  
  if (x>3) return (0)  
  if (x<0) return (0)  
  A = 1/(1-exp(-(3/lambda)^k))  
  pdf = A*(k/lambda)*(x/lambda)^(k-1)*exp(-(x/lambda)^k)  
  return (pdf)  
}
```

d)

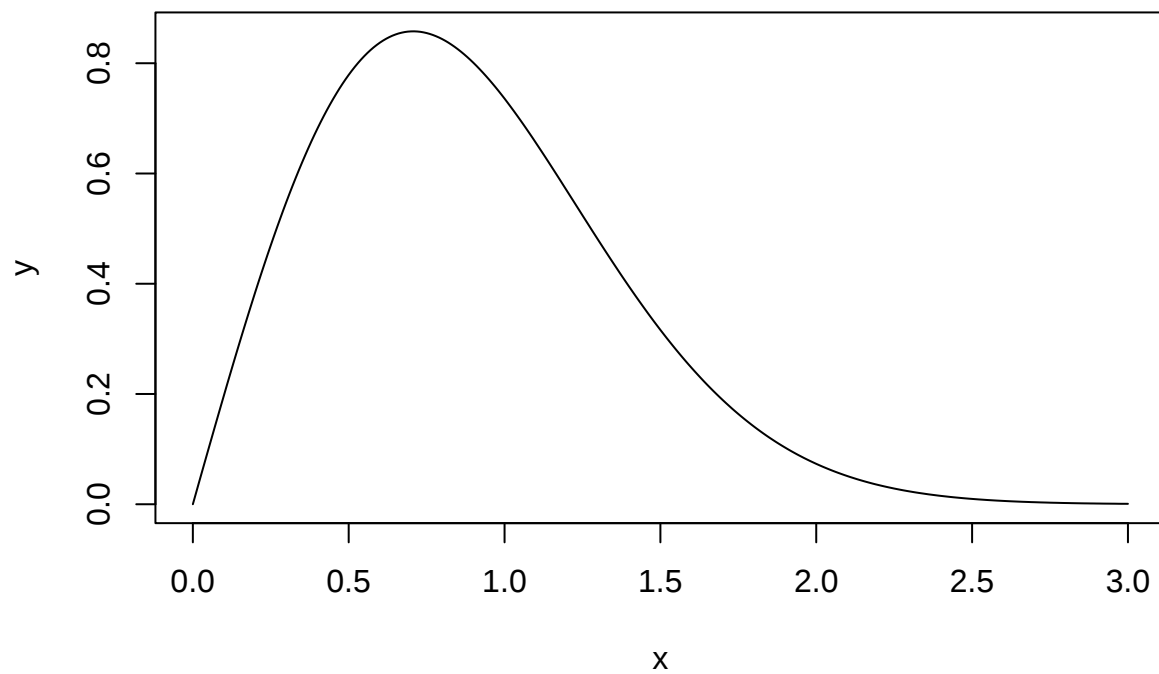
$k = 1, \lambda = 1.$

```
x = seq(0,3,0.01)
y = sapply(x, truncatedweibullpdf,k=1,lambda=1)
plot(x,y,type='l')
```



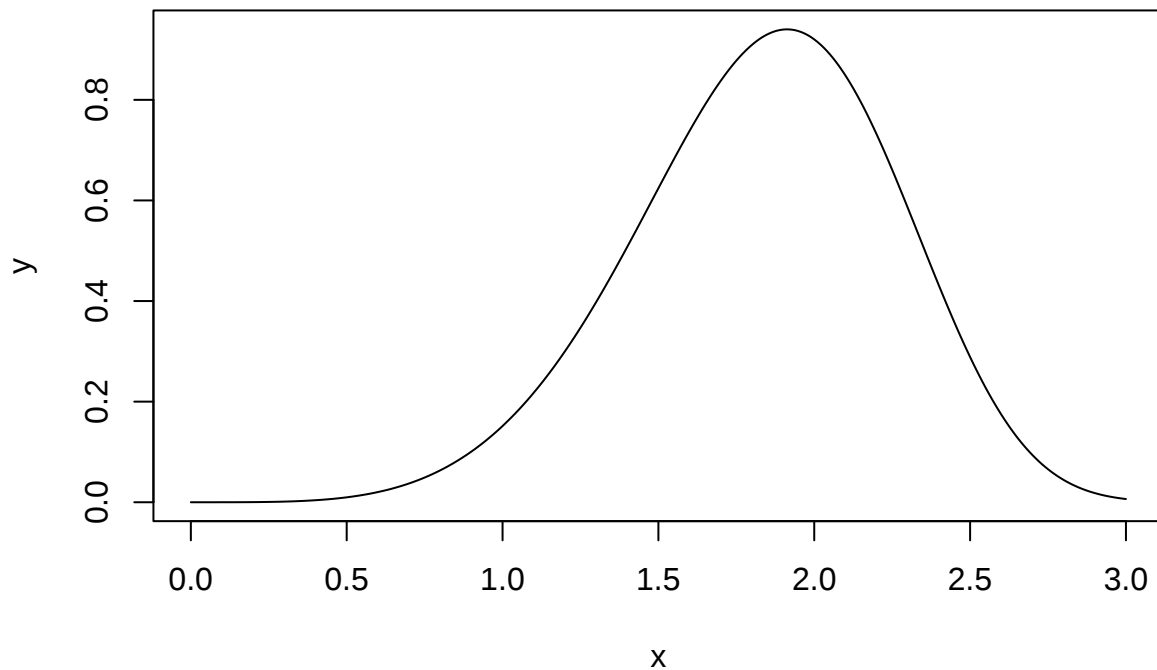
$k = 2, \lambda = 1.$

```
x = seq(0,3,0.01)
y = sapply(x, truncatedweibullpdf,k=2,lambda=1)
plot(x,y,type='l')
```



$k = 5, \lambda = 2.$

```
x = seq(0,3,0.01)
y = sapply(x, truncatedweibullpdf,k=5,lambda=2)
plot(x,y,type='l')
```



(e)

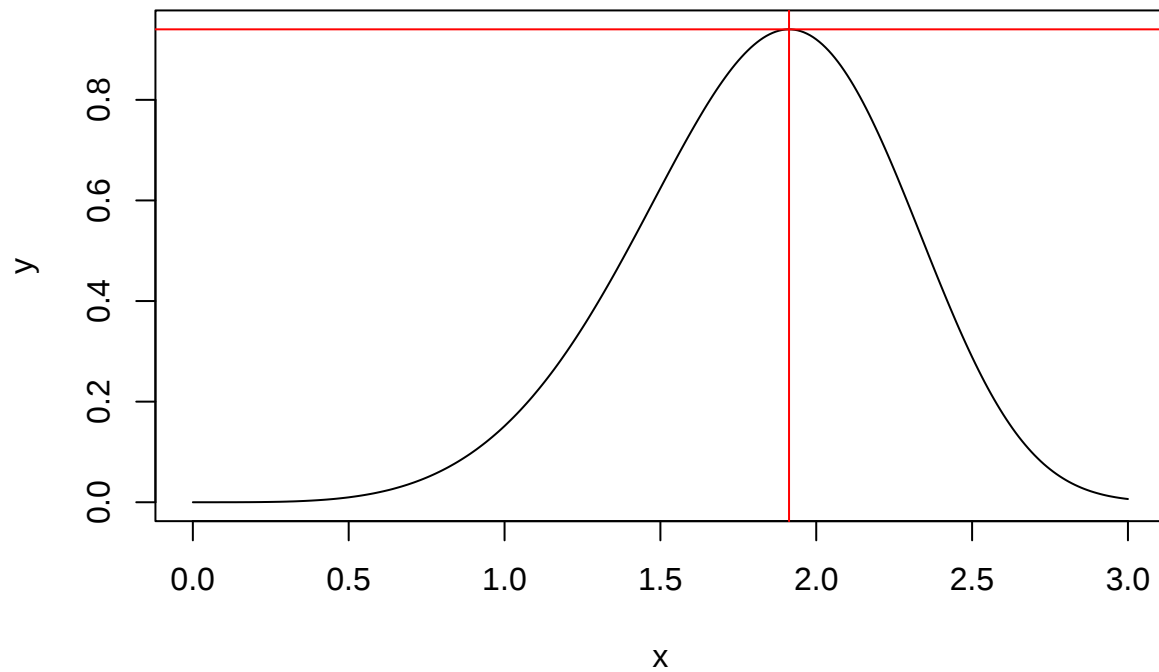
```
uniform2d <- function(n,Mx,My){
  x = runif(n,0,Mx)
  y = runif(n,0,My)
  res = matrix(c(x,y),ncol=2)
  return (res)
}
```

(f)

```
f <- function(x) truncatedweibullpdf(x,5,2)
optimize(f,lower=0,upper=3,maximum = TRUE)
```

```
## $maximum
## [1] 1.912714
##
## $objective
## [1] 0.9401457
```

```
x = seq(0,3,0.01)
y = sapply(x, truncatedweibullpdf,k=5,lambda=2)
plot(x,y,type='l')
abline(v=1.912714, col='red')
abline(h= 0.9401457, col='red')
```

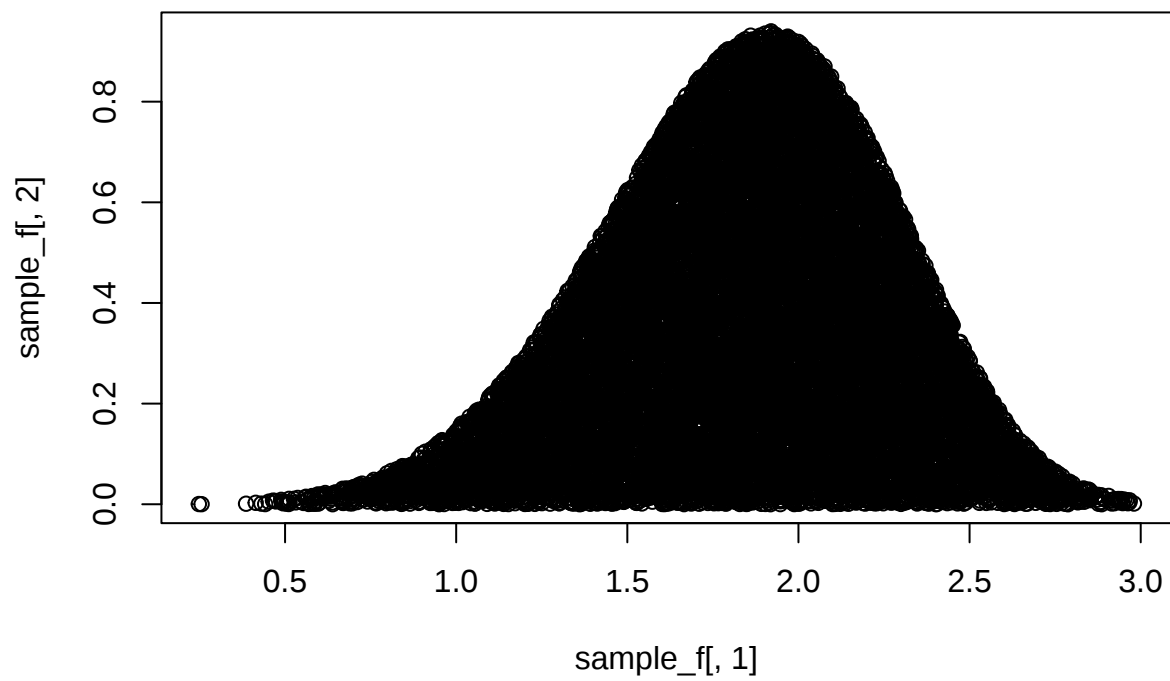


g)

```
rejection <- function(cordinates){
  index = rep(TRUE,nrow(cordinates))
  for ( i in 1:nrow(cordinates)){
    if (cordinates[i,2]>truncatedweibullpdf(cordinates[i,1],5,2)){
      index[i]=FALSE
    }
  }
  return(cordinates[index,])
}
```

f)

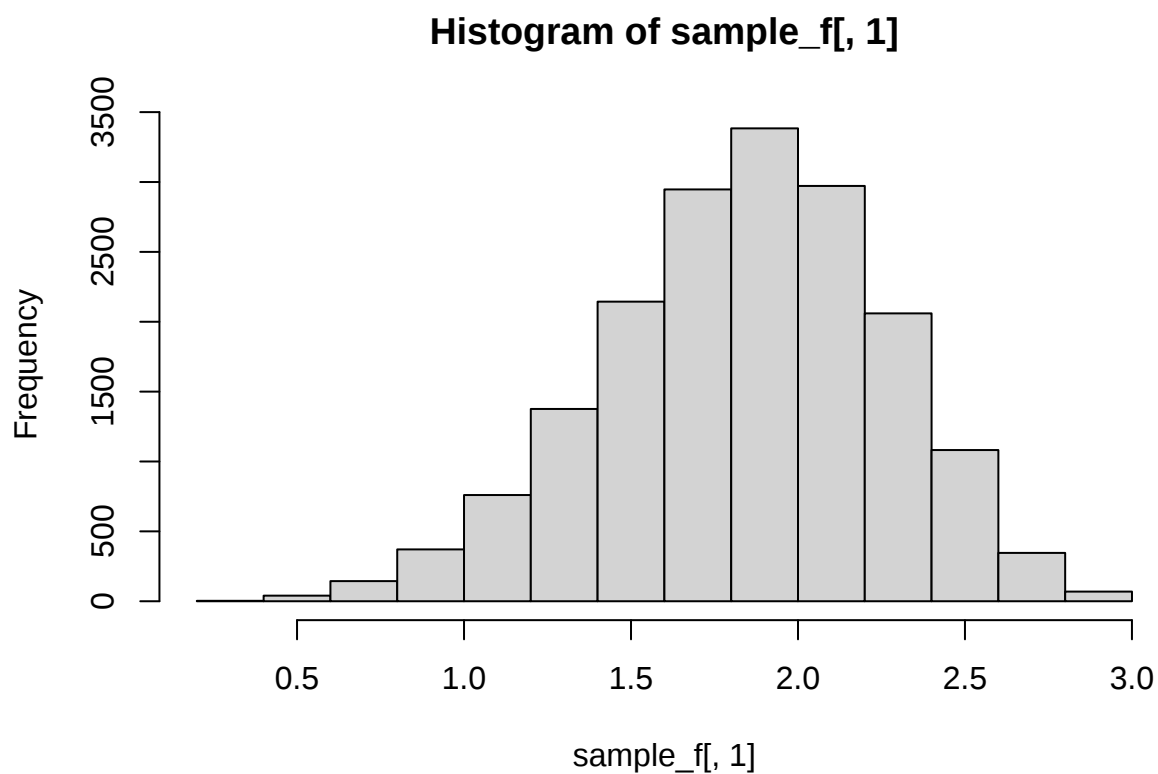
```
M = 0.9401457
propose_sample <- uniform2d(50000,3,M)
sample_f <- rejection(propose_sample)
plot(sample_f[,1],sample_f[,2])
```



This looks like the pdf of the truncated weibull distribution.

(i)

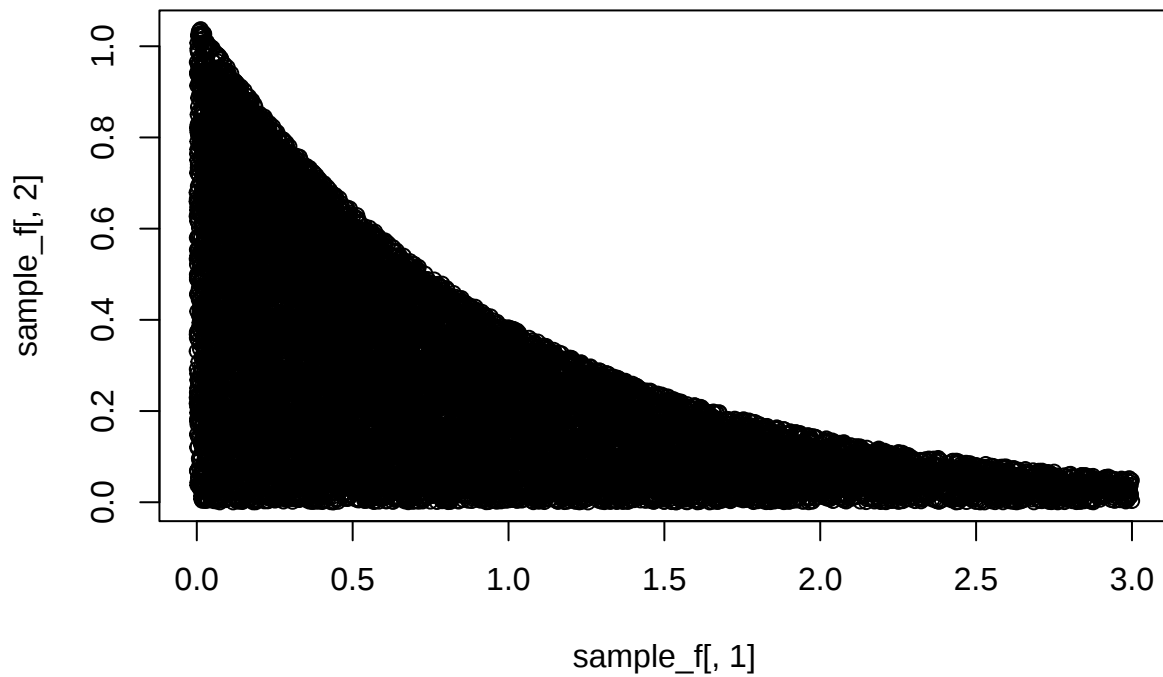
```
hist(sample_f[,1])
```



The histogram looks like the Weibull distribution.

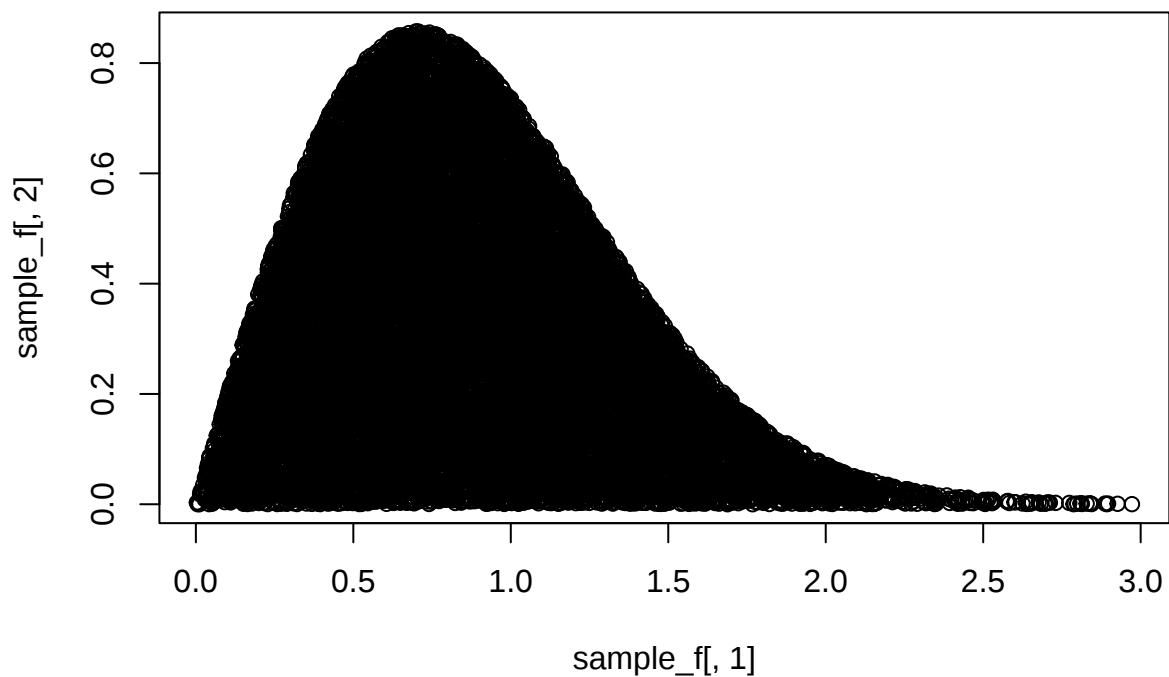
(i) $k = 1, \lambda = 1$.

```
f <- function(x) truncatedweibullpdf(x,1,1)
M = optimize(f,lower=0,upper=3,maximum = TRUE)$objective
propose_sample <- uniform2d(50000,3,M)
rejection <- function(cordinates){
  index = rep(TRUE,nrow(cordinates))
  for ( i in 1:nrow(cordinates)){
    if (cordinates[i,2]>truncatedweibullpdf(cordinates[i,1],1,1)){
      index[i]=FALSE
    }
  }
  return(cordinates[index,])
}
sample_f <- rejection(propose_sample)
plot(sample_f[,1],sample_f[,2])
```



$k = 2, \lambda = 1.$

```
f <- function(x) truncatedweibullpdf(x,2,1)
M = optimize(f,lower=0,upper=3,maximum = TRUE)$objective
propose_sample <- uniform2d(50000,3,M)
rejection <- function(cordinates){
  index = rep(TRUE,nrow(cordinates))
  for ( i in 1:nrow(cordinates)){
    if (cordinates[i,2]>truncatedweibullpdf(cordinates[i,1],2,1)){
      index[i]=FALSE
    }
  }
  return(cordinates[index,])
}
sample_f <- rejection(propose_sample)
plot(sample_f[,1],sample_f[,2])
```



2. Cauchy distribution

a)

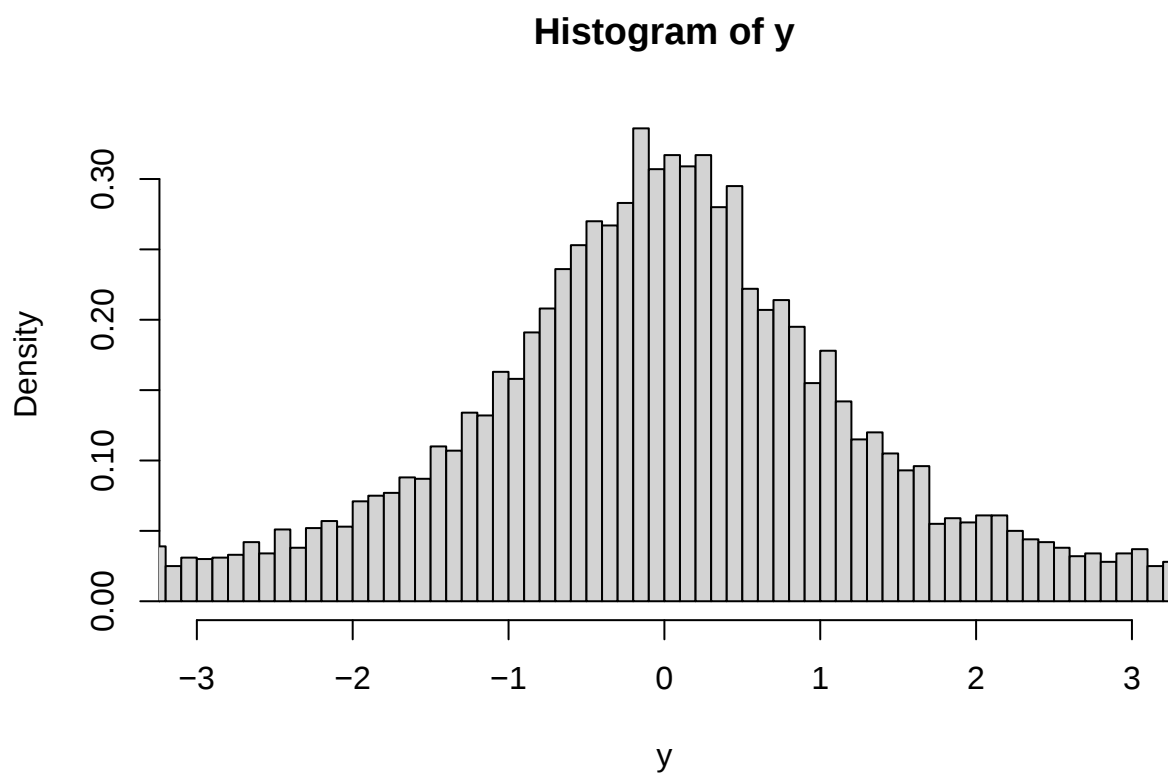
$$\begin{aligned}
 F_X(x) &= \frac{1}{\pi} \int_{-\infty}^x \frac{1}{1+t^2} dt \\
 &= \frac{1}{\pi} (\arctan x - \arctan(-\infty)) \\
 &= \frac{1}{\pi} \arctan x + \frac{1}{2}.
 \end{aligned}$$

$$F_X^{-1}(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right), x \in (0, 1).$$

b) Sample from n values from the uniform distribution, and then apply F_X^{-1} to these values.

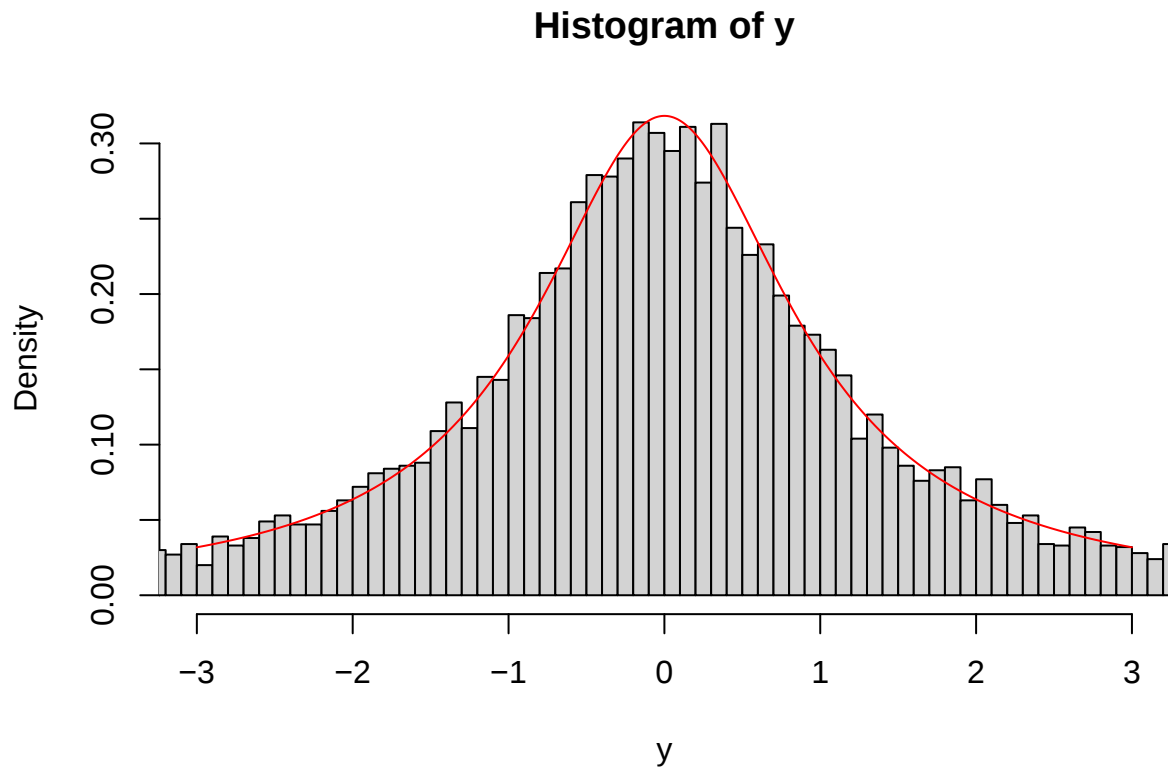
c)

```
x = runif(10000,0,1)
y = tan(pi*(x-1/2))
hist(y,freq = FALSE,xlim=c(-3,3),breaks =c(min(y),seq(-5,5,0.1),max(y)))
```



(d)

```
x = runif(10000,0,1)
y = tan(pi*(x-1/2))
hist(y,freq = FALSE,xlim=c(-3,3),breaks =c(min(y),seq(-5,5,0.1),max(y)))
lines(seq(-3,3,0.01),dcauchy(seq(-3,3,0.01)),col='red')
```



(e) Let $X \sim U(0, 1)$, then $Y = 2X + 5 \sim U(5, 7)$. Note that,

$$\int_5^7 f_X(x) dx = 2 \int_5^7 f_X(x) \frac{1}{2} dx = 2E(f_X(Y)).$$

```
x = runif(20000,0,1)
x = 2*x+5
f <- function(x) 1/(pi*(1+x^2))
mean(f(x))*2
```

```
## [1] 0.01769587
```

3. Cauchy distribution and Rejection sampling

a) Cauchy distribution can take any values in $(-\infty, \infty)$. Note that,

$$\frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \leq M \frac{1}{\pi(1+x^2)}$$

Thus,

$$M \geq \sqrt{\pi/2} (1+x^2) \exp(-x^2/2)$$

```
M = optimize(function(x) sqrt(pi/2)*(1+x^2)*exp(-x^2/2), c(0,10),maximum = TRUE)$objective
print(M)
```

```
## [1] 1.520347
```

```
u = runif(10000)
y = tan(pi*(runif(10000,0,1)-1/2))
```

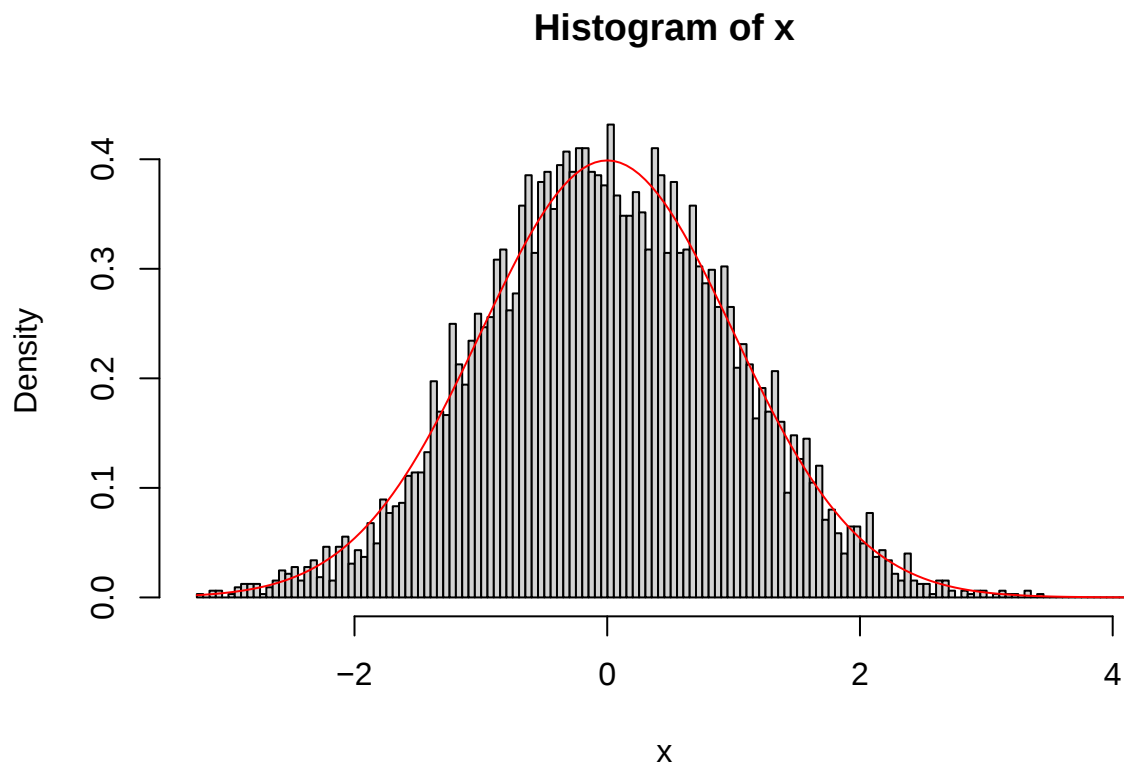
We accept the sample if

$$u < \frac{1}{M} \sqrt{\pi/2} (1 + y^2) \exp(-y^2/2)$$

(c)

```
u = runif(10000)
y = tan(pi*(runif(10000,0,1)-1/2))
index = (u < sqrt(pi/2)*(1+y^2)*exp(-y^2/2)/M)
x = y[index]
```

```
hist(x,breaks = seq(min(x)-0.05,max(x)+0.05,0.05),freq = FALSE)
lines(seq(min(x)-0.05,max(x)+0.05,0.05),dnorm(seq(min(x)-0.05,max(x)+0.05,0.05)),col='red')
```



d) The accepted ratio is

```
sum(index)/10000
```

```
## [1] 0.6486
```

1/M is

```
1/M
```

```
## [1] 0.6577446
```

They are close.