(i) $f(x) = \frac{x-1}{3(x+2)(x+1)}$

$= \frac{x+1-2}{3(x+2)(x+1)}$

$= \frac{1}{3(x+2)} - \frac{2}{3(x+2)(x+1)}$

$= \frac{1}{3(x+2)} - \frac{2}{3}\left(\frac{1}{x+1} - \frac{1}{x+2}\right)$

$= \frac{1}{x+2} - \frac{2}{3} \cdot \frac{1}{x+1}.$

(ii)

$$\frac{dy}{dx} + f(x)\, y = \frac{2}{3} \frac{1}{x+1}.$$

The integrating factor is

$$\mu(x) = e^{\int f(x)\, dx}.$$

Note that

$$\int f(x)\,dx = \int \frac{1}{x+2} - \frac{2}{3}\frac{1}{x+1}\,dx$$

$$= \log(x+2) - \frac{2}{3}\log(x+1).$$

Thus

$$\mu(x) = e^{\int f(x)\,dx} = \frac{x+2}{(x+1)^{2/3}}.$$

let $g(x) = \frac{2}{3}\frac{1}{x+1}$

(iii) The general soultion is

$$y = \frac{c}{\mu(x)} + \frac{\int \mu(x)\,g(x)\,dx}{\mu(x)}.$$

Note that $\int \mu(x)\,g(x) = \frac{2}{3}\int \frac{x+2}{(x+1)^{\frac{5}{3}}}\,dx$

Thus

$$y = c\cdot\frac{(x+1)^{\frac{2}{3}}}{x+2} + \frac{2}{3}\cdot\frac{(x+1)^{\frac{2}{3}}}{x+2}\int \frac{x+2}{(x+1)^{\frac{5}{3}}}\,dx$$

(iv)

Let $u=x+1$

$$\int \frac{x+2}{(x+1)^{\frac{5}{3}}}\,dx = \int \frac{u+1}{u^{\frac{5}{3}}}\,du$$

$$= \int \left(u^{-\frac{2}{3}} + u^{-\frac{5}{3}}\right)du$$

$$= 3u^{\frac{1}{3}} - \frac{3}{2}u^{-\frac{2}{3}}$$

$$= 3(x+1)^{\frac{1}{3}} - \frac{3}{2}(x+1)^{-\frac{2}{3}}.$$

Thus

$$y = C\cdot\frac{(x+1)^{\frac{2}{3}}}{x+2} + \frac{2}{3}\cdot\frac{(x+1)^{\frac{2}{3}}}{x+2}\cdot\left(3(x+1)^{\frac{1}{3}} - \frac{3}{2}(x+1)^{-\frac{2}{3}}\right)$$

$$= C\cdot\frac{(x+1)^{\frac{2}{3}}}{x+2} + 2\cdot\frac{x+1}{x+2} - \frac{1}{x+2}.$$

$$= C\cdot\frac{(x+1)^{\frac{2}{3}}}{x+2} + \frac{2x+1}{x+2}$$

(v)

$y(0) = 0$

Thus $\frac{c}{2} + \frac{1}{2} = 0 \quad c = -\frac{1}{2}$

$$y = -\frac{1}{2} \frac{(x+1)^{\frac{2}{3}}}{x+2} + \frac{2x+1}{x+2}.$$