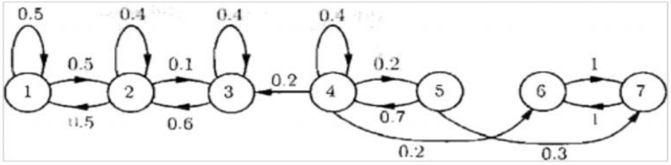
Consider the Markov chain shown below:



[7 points] Assume that the process starts in state 1 but we begin observe it after it reaches steady-state.

i. Find the probability that the state increases by one during the first transition that we observed.

ii. Given that the state increased by one during the first transition that we observed, find the conditional probability that when we started observing, the process was in state 2 and the state increased by one.

iii. *During the first change of state* that we observed, find the probability that the state increased by one.

[6 points] Assume that the process starts in state 4.

i. For each recurrent class, determine the probability that we eventually reach that class.

ii. What is the expected number of transitions up to and including the transition at which we reach a recurrent state for the first time?

Answer:

Since state 1, 2, 3 is a recurrent class, hence when the process starts in state 1, we only consider the steady-state of state 1,2 and 3 in this Markov chain (when the process starts in state 1, the transitions cannot pass state 4-7)

The state transition probability matrix between state 1, 2 and 3 is $\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.4 & 0.1 \\ 0 & 0.6 & 0.4 \end{bmatrix}$.

The balance equations of steady-state are:

$$\begin{cases} \pi_1 = 0.5\pi_1 + 0.5\pi_2 + 0\pi_3 \\ \pi_2 = 0.5\pi_1 + 0.4\pi_2 + 0.6\pi_3 \\ \pi_3 = 0\pi_1 + 0.1\pi_2 + 0.4\pi_3 \end{cases}$$

And since $\pi_1 + \pi_2 + \pi_3 = 1$, hence $\pi_1 = \pi_2 = \frac{6}{13}$, $\pi_3 = \frac{1}{13}$.

Hence when the Markov chain reaches steady-state, the steady-state probabilities are $\pi_1 = \pi_2 = \frac{6}{13}$ and $\pi_3 = \frac{1}{13}$.

i. let the probability that the state increases by one during the first transition that we observed is P_1.

the state increases by one during the first transition that we observed contains two situations: $state1 \to state2$ and $state2 \to state3$.

The probability of $state1 \to state2$ is $\pi_1 \times p_{12} = \frac{6}{13} \times 0.5 = \frac{3}{13}$;

Similarly, the probability of $state2 \to state3$ is $\pi_2 \times p_{23} = \frac{6}{13} \times 0.1 = \frac{3}{65}$;

Hence $P_1 = \pi_1 \times p_{12} + \pi_2 \times p_{23} = \frac{3}{13} + \frac{3}{65} = \frac{18}{65}$.

ii. let the conditional probability is P_2.

$$P_2 = \frac{\pi_2 \times p_{23}}{\pi_1 \times p_{12} + \pi_2 \times p_{23}} = \frac{\pi_2 \times p_{23}}{P_1} = \frac{\frac{3}{65}}{\frac{18}{65}} = \frac{1}{6}.$$

iii. The first change of state contains: $state1 \to state2$, $state2 \to state1$,

$state2 \to state3$ and $state3 \to state2$.

The state increases by one contains: $state1 \to state2$ and $state2 \to state3$.

Let during the first change of state that we observed, the probability of the state increases by one is P_3.

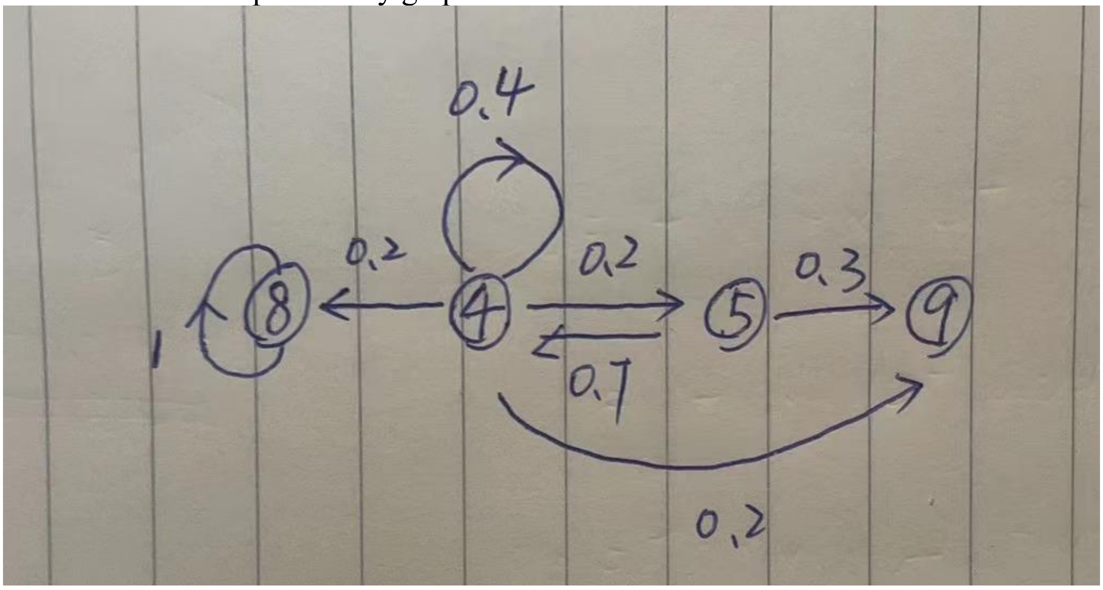
Then $P_3 = \dfrac{\pi_1 \times p_{12} + \pi_2 \times p_{23}}{\pi_1 \times p_{12} + \pi_2 \times (p_{21} + p_{23}) + \pi_3 \times p_{32}} = \dfrac{\frac{6}{13} \times 0.5 + \frac{6}{13} \times 0.1}{\frac{6}{13} \times 0.5 + \frac{6}{13} \times 0.6 + \frac{1}{13} \times 0.6} = \dfrac{1}{2}$.

Assume that the process starts in state 4.

The whole Markov chain contains two recurrent classes: $\{1,2,3\}$ and $\{6,7\}$.

To simplify this problem, we set $\{1,2,3\}$ to state 8 and set $\{6,7\}$ to state 9.

Then the transition probability graph of Markov chain can be redrawn as:



i. for the recurrent class $\{1,2,3\}$, also state 8,

let the probability a_i is the probability that from state i eventually reach state 8 and according to the above probability graph, they satisfy:

$$\begin{cases} a_4 = 0.2a_8 + 0.4a_4 + 0.2a_5 + 0.2a_9 \\ a_5 = 0.7a_4 + 0.3a_9 \\ a_8 = 1 \\ a_9 = 0 \end{cases} \Rightarrow a_4 = \frac{10}{23}$$

Hence the probability that from state 4 eventually reach state 8 is $a_4 = \dfrac{10}{23}$.

Similarly, let the probability b_i is the probability that from state i eventually reach state 9 and according to the above probability graph, they satisfy:

$$\begin{cases} b_4 = 0.2b_8 + 0.4b_4 + 0.2b_5 + 0.2b_9 \\ b_5 = 0.7b_4 + 0.3b_9 \\ b_8 = 0 \\ b_9 = 1 \end{cases} \Rightarrow b_4 = \frac{13}{23}$$

Hence the probability that from state 4 eventually reach state 9 is $b_4 = \frac{13}{23}$.

ii. Mean first passage times is that for any state to reach a specific recurrent state for the first time:

$$t_i = E[number\ of\ transitions\ to\ reach\ s\ for\ the\ first\ time,\ starting\ from\ i]$$
$$= E[min\{n \geq 0 | X_n = s\} | X_0 = i]$$

$$\begin{cases} t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j, & for\ all\ i \neq s \\ t_s = 0 \end{cases}$$

Hence let the expected number of transitions up to and including the transition at which we reach the recurrent state 8 for the first time is u_i then

$$\begin{cases} u_4 = 1 + 0.2u_8 + 0.4u_4 + 0.2u_5 \\ u_5 = 1 + 0.7u_4 \\ u_8 = 0 \end{cases} \Rightarrow u_4 = \frac{60}{23}$$

let the expected number of transitions up to and including the transition at which we reach the recurrent state 9 for the first time is v_i then

$$\begin{cases} v_4 = 1 + 0.4v_4 + 0.2v_5 + 0.2v_9 \\ v_5 = 1 + 0.7v_4 + 0.3v_9 \\ v_9 = 0 \end{cases} \Rightarrow v_4 = \frac{60}{23}$$

Hence the expected number of transitions up to and including the transition at which we reach both the recurrent state 8 and the recurrent state 9 for the first time are $\frac{60}{23}$.