Ques 33454

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(We assume f is continuous, otherwise $\max\{f(x) : x \in \mathrm{ext}(C)\}$ may not exist)

C is a compact subset in Euclidean space, and f is continuous, so $\max\{f(x) : x \in C\} = f(p)$ for some point $p \in C$. By Krein-Milman theorem[1], a compact convex subset of an Euclidean space is equal to the convex hull of its extreme points. So $C = \mathrm{conv}(\mathrm{ext}(C))$, and $p = \sum_{i=1}^n \alpha_i t_i$ for some $\alpha_i > 0, \sum_{i=1}^n \alpha_i = 1, t_i \in \mathrm{ext}(C)$.

Now f is convex, so $f\left(\sum_{i=1}^n \alpha_i t_i\right) \leqslant \sum_{i=1}^n \alpha_i f(t_i)$, then

$$f(p) = f\left(\sum_{i=1}^n \alpha_i t_i\right) \leqslant \sum_{i=1}^n \alpha_i f(t_i) \leqslant \max_{1\leqslant i\leqslant n} f(t_i) \leqslant \max\{f(x) : x \in C\} = f(p)$$

So $\leqslant$ must all be $=$. So $f(t_i) = f(p)$, $i = 1, \cdots, n$. Then

$$\max\{f(x) : x \in C\} = f(p) = f(t_i) \leqslant \max\{f(x) : x \in \mathrm{ext}(C)\} \leqslant \max\{f(x) : x \in C\}$$

So

$$\max\{f(x) : x \in C\} = \max\{f(x) : x \in \mathrm{ext}(C)\}$$

Example: $C = \left\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leqslant 1\right\}$, $f(x) = x_1^2 + \cdots + x_n^2$ is a non-constant convex function. Then $\mathrm{ext}(C) = \left\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\right\}$, and $\max\{f(x) : x \in C\} = f(y)$, $\forall y \in \mathrm{ext}(C)$.

[1] To be precise, Ernst Steinitz's theorem ("Bedingt konvergente Reihen und konvexe Systeme VI, VII", J. Reine Angew. Math., 146: 1–52;)