Ques 32120

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(i)

By definition, we have $v(\varnothing) = 0$.

If A, B are subsets of $\mathbb{R}^n$ such that $A \subseteq B$: when $A = \varnothing$, we know $v(B) = 0$ or 1, so $v(B) \geqslant 0 = v(A)$; when $A \neq \varnothing$, then $A \subseteq B$ implies $B \neq \varnothing$, so by definition $v(B) = 1 = v(A)$. To conclude, we always have $v(A) \leqslant v(B)$.

For arbitrary subsets $A_1, A_2, \cdots$ of $\mathbb{R}^n$: if $\{A_i\}$ are not all empty, suppose A_{i_0} is non empty, then $A = \bigcup_{i=1}^{\infty} A_i$ is also non-empty, we have $v(A) = 1 = v(A_{i_0}) \leqslant \sum_{i=1}^{\infty} v(A_i)$; if $\{A_i\}$ are all empty, then $A = \bigcup_{i=1}^{\infty} A_i$ is also empty, we have $v(A) = 0 = \sum_{i=1}^{\infty} 0 = \sum_{i=1}^{\infty} v(A_i)$. To conclude, we always have $v\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} v(A_i)$.

So v is an outer measure.

(ii)

By definition, a subset E of $\mathbb{R}^n$ is v-measurable, if and only if for any subset A of $\mathbb{R}^n$, we have $v(A) = v(A \cap E) + v(A \setminus E)$.

When $E = \varnothing$, then $v(A \cap \varnothing) + v(A \setminus \varnothing) = v(\varnothing) + v(A) = 0 + v(A) = v(A)$. So $\varnothing$ is v-measurable.

When $E = \mathbb{R}^n$, then $v(A \cap \mathbb{R}^n) + v(A \setminus \mathbb{R}^n) = v(A) + v(\varnothing) = v(A) + 0 = v(A)$. So $\mathbb{R}^n$ is v-measurable.

When $E \neq \varnothing, E \neq \mathbb{R}^n$. Take $A = \mathbb{R}^n$, then $\mathbb{R}^n \cap E \neq \varnothing$, $\mathbb{R}^n - E \neq \varnothing$, so $v(\mathbb{R}^n \cap E) + v(\mathbb{R}^n - E) = 1 + 1 = 2 \neq 1 = v(\mathbb{R}^n)$. So E is not v-measurable.

So all v-measurable sets are $\varnothing$ and $\mathbb{R}^n$.