Ques 31152

Honghao Chen, 2842453234@qq.com

2022.05.02

Consider the counter C_R: a line segament from $-R$ to R, then a semicircle $Re^{\mathrm{i}\theta}$ ($\theta \in [0, \pi]$).

$f(x) = \frac{x^2}{(x^2+16)^2}$ have two poles $\pm 4\mathrm{i}$ and they are 2-order poles. $+4\mathrm{i}$ is inside the counter C_R and the residue is $\operatorname{Res}_{z=4\mathrm{i}} f(z) = \lim_{z\to 4\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} \left((z-4\mathrm{i})^2 f(z)\right) = -\frac{1}{16}\mathrm{i}$.

By Cauchy Residue theorem,

$$\int_{C_R} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\operatorname{Res}_{z=4\mathrm{i}} f(z) = \frac{\pi}{8}.$$

Now on the semicircle, when R is large enough,

$$\begin{aligned}
\left|\int_{\{|z|=R,\operatorname{Im}(z)\geqslant 0\}} f(x)\,\mathrm{d}x\right| &= \left|\int_0^\pi \frac{R^2e^{2\mathrm{i}\theta}}{(R^2e^{2\mathrm{i}\theta}+16)^2} R\mathrm{i}\theta\,\mathrm{d}\theta\right| \\
&\leqslant \int_0^\pi \frac{R^3\theta}{\left|R^2e^{2\mathrm{i}\theta}+16\right|^2}\,\mathrm{d}\theta \\
&\leqslant \int_0^\pi \frac{R^3\theta}{\left(\frac{1}{2}R^2\right)^2}\,\mathrm{d}\theta \quad (*) \\
&= \frac{2\pi^2}{R} \to 0
\end{aligned}$$

Here, $(*)$ holds because, when $R^2 > 16$ we have $\left|R^2e^{2\mathrm{i}\theta}+16\right| \geqslant R^2 - 16$ by triangle inequality, and $R^2 - 16 \geqslant \frac{1}{2}R^2$ when $R^2 \geqslant 32$.

So $\int_{\{|z|=R,\operatorname{Im}(z)\geqslant 0\}} f(x)\,\mathrm{d}x \to 0$ as $R \to \infty$.

Now

$$\frac{\pi}{8} = \int_{C_R} f(x)\,\mathrm{d}x = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{\{|z|=R,\operatorname{Im}(z)\geqslant 0\}} f(x)\,\mathrm{d}x,$$

take the limit $\lim_{R\to\infty}$ on both sides, we get $\frac{\pi}{8} = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+16)^2}\,\mathrm{d}x$.