Ques 31141

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Given a Jordan block J_λ^n of size $n \times n$, we know $\operatorname{rank}(J_\lambda^n - \lambda I)^m = \begin{cases} n-m & ,m \leqslant n \\ 0 & ,m > n \end{cases}$.

If the Jordan form of B has n_i Jordan blocks of size $i \times i$. Then

$$\operatorname{rank} I = n_1 \cdot 1 + n_2 \cdot 2 + n_3 \cdot 3 + n_4 \cdot 4 + \cdots$$

$$\operatorname{rank}(B+I) = n_1 \cdot 0 + n_2 \cdot 1 + n_3 \cdot 2 + n_4 \cdot 3 + \cdots$$

$$\operatorname{rank}(B+I)^2 = n_1 \cdot 0 + n_2 \cdot 0 + n_3 \cdot 1 + n_4 \cdot 2 + \cdots$$

$$\operatorname{rank}(B+I)^3 = n_1 \cdot 0 + n_2 \cdot 0 + n_3 \cdot 0 + n_4 \cdot 1 + \cdots$$

So

$$n_1 + n_2 + n_3 + n_4 + \cdots = \operatorname{rank} I - \operatorname{rank}(B+I)$$

$$n_2 + n_3 + n_4 + \cdots = \operatorname{rank}(B+I) - \operatorname{rank}(B+I)^2$$

$$n_3 + n_4 + \cdots = \operatorname{rank}(B+I)^2 - \operatorname{rank}(B+I)^3$$

So

$$n_i = \operatorname{rank}(B+I)^{i-1} - 2\operatorname{rank}(B+I)^i + \operatorname{rank}(B+I)^{i+1}.$$

Now $\operatorname{rank} I = 5$, $\operatorname{rank}(B+I) = 3$, $\operatorname{rank}(B+I)^2 = 1$. By direct calculation, $\operatorname{rref}(B+I) \cdot (B+I)^2 = 0$, so $(B+I)^3 = 0$.

So we get $n_1 = 0$, $n_2 = 1$, $n_3 = 1$, $n_4 = 0$, $n_5 = 0$.

So the Jordan form of B is $\begin{pmatrix} -1 & 1 & & & \\ & -1 & & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}$.

Let $e_5 = (1,0,0,0,0)^t$, we know $(B+I)^2 e_5 \neq 0$. Let $e_4 = (B+I)e_5 = (0,-1,-1,-1,-1)^t$, $e_3 = (B+I)^2 e_5 = (1,-1,1,0,0)^t$.

Let $e_2 = (0,0,0,2,1)^t$, $e_1 = (B+I)e_2 = (9,6,-1,10,-5)^t$. We know $(B+I)^2 e_2 = 0$.

By direct verification, $\beta = \{e_1, e_2, e_3, e_4, e_5\}$ is a basis. By our construction, $(L_B)_\beta^\beta$ is of the form $\begin{pmatrix} -1 & 1 & & & \\ & -1 & & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}$.

We have

$$((L_B)_\beta^\beta)^\ell = \begin{pmatrix} (-1)^\ell & \ell(-1)^{\ell-1} & & & \\ & (-1)^\ell & & & \\ & & (-1)^\ell & \ell(-1)^{\ell-1} & \frac{\ell(\ell-1)}{2}(-1)^{\ell-2} \\ & & & (-1)^\ell & \ell(-1)^{\ell-1} \\ & & & & (-1)^\ell \end{pmatrix}.$$

For $z = \sum_{i=1}^5 z_i e_i$, to make $\left\{B^\ell z\right\}_{\ell \geqslant 1}$ bounded, we must require the first, the 4-th coordinate (under β) of $B^\ell z$ to be bounded, so $z_2 = z_5 = 0$. Then 3-rd coordinate bounded forces $z_4 = 0$. Now for $z = z_1 e_1 + z_3 e_3$, we have $B^\ell z = (-1)^\ell z$, is bounded. So z should live in $\operatorname{Span}\{e_1, e_3\} = \operatorname{Nullspace}(B + I) = \operatorname{Span}\{(1, -1, 1, 0, 0)^t, (0, 3, -2, 2, -1)^t\}$ to make $\left\{B^\ell z\right\}$ bounded.